

The Hilbert Series of Adjoint SQCD

Amihay Hanany, Noppadol Mekareeya and Giuseppe Torri

Theoretical Physics Group, The Blackett Laboratory

Imperial College London, Prince Consort Road

London, SW7 2AZ, UK

Email: a.hanany, n.mekareeya07, giuseppe.torri08@imperial.ac.uk

ABSTRACT: We use the plethystic exponential and the Molien–Weyl formula to compute the Hilbert series (generating functions), which count gauge invariant operators in $\mathcal{N} = 1$ supersymmetric $SU(N_c)$, $Sp(N_c)$, $SO(N_c)$ and G_2 gauge theories with 1 adjoint chiral superfield, fundamental chiral superfields, and zero classical superpotential. The structure of the chiral ring through the generators and relations between them is examined using the plethystic logarithm and the character expansion technique. The palindromic numerator in the Hilbert series implies that the classical moduli space of adjoint SQCD is an affine Calabi–Yau cone over a weighted projective variety.

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1. Introduction and Summary

The rich structures of $\mathcal{N} = 1$ Supersymmetric Quantum Chromodynamics (SQCD) chiral rings and moduli spaces has been studied in [1, 2] using the Plethystic Programme [3, 4, 5, 6, 7, 8, 9, 10], Molien–Weyl formula [11, 12, 13, 14, 15], and character expansions [9, 10, 16]. We can generalise this theory in an interesting way by adding to it a chiral superfield ϕ in the adjoint representation, and use the name termed in the literature, $\mathcal{N} = 1$ **adjoint SQCD**. In this paper, we shall focus on the theory with the $SU(N_c)$, $Sp(N_c)$, $SO(N_c)$ and G_2 gauge groups with vanishing classical superpotential.

There have been a series of works [17, 18, 19, 20, 21, 22] on the $SU(N_c)$ adjoint SQCD, as well as [23, 24] on the $SO(N_c)$ and $Sp(N_c)$ theories with various classical superpotentials. It is known that the classical moduli space of the $SU(N_c)$ theory does not get quantum corrections [19, 20, 21]. However, due to technical difficulties, many aspects (e.g., Seiberg duality) of the zero classical superpotential theories have yet to be fully understood¹. The main aim of this paper is to examine the structure of the chiral rings of adjoint SQCD (with zero superpotential) through the generators of the gauge invariant operators (GIOs) and their relations.

We use the plethystic exponential and the Molien–Weyl formula to obtain Hilbert series², which count GIOs. The generators and the relations in the chiral ring can be extracted from the plethystic logarithm of the Hilbert series. Using the character expansion technique, we can also figure out how these generators and relations transform under the global symmetry.

¹Regarding this, let us quote the authors of [18]: ‘This interesting model has so far resisted all attempts at a detailed understanding.’

²There are 3 words which are synonymous: partition function, generating function and Hilbert series. The first one is the physics literature name, whereas the second and third ones typically appear in the mathematical literature.

Hilbert series also contain information about geometrical properties of the moduli space. We shall see in subsequent sections, for example, that plethystic logarithms of Hilbert series can indicate whether the moduli space is a complete intersection, and that the palindromic property of the Hilbert series implies that the moduli space is Calabi–Yau [1, 2, 16].

Below, we collect the main results of our work.

Outline and key points.

- In Section 2, we summarise the Plethystic Programme and Molien–Weyl formula.
- In Section 4, Hilbert series of adjoint SQCD with $SU(N_c)$ gauge group are constructed. We analyse the generators and relations using the plethystic logarithm. We find that the total number of generators in $SU(N_c)$ theory with N_f flavours is of order $N_f^{N_c}$.
 - In Subsection 4.2, we count adjoint baryons in the $SU(N_c)$ theory. This leads to an interesting combinatorial problem of partitions. We find an exponentially large number of adjoint baryons when N_c is large. The asymptotic formula is given by (4.46).
 - In Subsection 4.3, the canonical free energy of the theory is derived and found to be of order $N_f N_c$.
 - In Subsection 4.4, we focus on the complete intersection moduli space of the $SU(N_c)$ theory (*i.e.*, with 1 flavour). There is exactly one basic relation in this case. A general expression of such a relation for any N_c is given by (4.57).
- The structures of generators and relations of the $Sp(N_c)$, $SO(N_c)$ and G_2 adjoint SQCD are studied using Hilbert series in Sections 5, 6, and 7. It is found that $Sp(N_c)$ theories with 1 flavor, and $SO(N_c)$ theories with 2 flavors have a moduli space which is a complete intersection, while the moduli space of $SO(N_c)$ theories with 1 flavor is freely generated. The number of relations for the complete intersection cases is equal to the rank of the gauge group and not to 1 as in the case of $SU(N_c)$ with 1 flavor.
- In Section 8, we take a geometric aperçu of the moduli space of adjoint SQCD. We establish that the classical moduli space is an irreducible affine Calabi–Yau cone.

Notation for representations. In this paper, we represent an irreducible representation of a group G by its highest weight $[a_1, \dots, a_r]$, where $r = \text{rank } G$. Young diagrams are also used in order to avoid cluttered notation. We also slightly abuse terminology by referring to each character by its corresponding representation.

2. Plethystic Programme and Molien–Weyl Formula: A Recapitulation

In order to write down explicit formulae and for performing computations we need to introduce weights for the different elements in the maximal torus of the different groups. We use

- z_a (with $a = 1, \dots, \text{rank } G$) to denote ‘colour’ weights, *i.e.* coordinates of the maximal torus of the gauge group G ;
- t_i (with $i = 1, \dots, N_f$) to denote ‘flavour’ weights for the fundamental chiral superfield Q , *i.e.* coordinates of the maximal torus of the global $U(N_f)_1$ symmetry ;
- \tilde{t}_i (with $i = 1, \dots, N_f$) to denote ‘flavour’ weights for the antifundamental chiral superfield \tilde{Q} , *i.e.* coordinates of the maximal torus of the global $U(N_f)_2$ symmetry ;
- s to count the chiral superfield ϕ in the adjoint representation.

These weights have the interpretation of fugacities for the charges they count and the characters of the representations are functions of these variables. Henceforth, we shall take t_i, \tilde{t}_i, s to be complex variables such that their absolute values lie between 0 and 1.

Below we summarise important facts and conventions of the gauge groups and their Lie algebras which we shall use later [25]:

The group $SU(N_c)$. Let us take the weights of the fundamental representation of $SU(N_c)$ to be

$$L_1 = (1, 0, \dots, 0) , \quad L_k = (0, 0, \dots, -1, 1, \dots, 0) , \quad L_{N_c} = (0, \dots, -1) , \quad (2.1)$$

where all L 's are $(N_c - 1)$ -tuples, and for L_k (with $2 \leq k \leq N_c - 1$), we have -1 in the $(k - 1)$ -th position and 1 in the k -th position. With this choice of weights, we find that the characters of the fundamental and antifundamental representations of $SU(N_c)$ are

$$\begin{aligned} [1, 0, \dots, 0]_{SU(N_c)}(z_a) &= z_1 + \sum_{k=2}^{N_c-1} \frac{z_k}{z_{k-1}} + \frac{1}{z_{N_c-1}} , \\ [0, \dots, 0, 1]_{SU(N_c)}(z_a) &= \frac{1}{z_1} + \sum_{k=2}^{N_c-1} \frac{z_{k-1}}{z_k} + z_{N_c-1} . \end{aligned} \quad (2.2)$$

The roots of the Lie algebra of $SU(N_c)$ are $\{L_a - L_b\}_{a \neq b}$. The character of the adjoint representation can be written as

$$\begin{aligned} [1, 0, \dots, 0, 1]_{SU(N_c)} &= [1, 0, \dots, 0]_{SU(N_c)} \times [0, \dots, 0, 1]_{SU(N_c)} - 1 \\ &= (N_c - 1) + \sum_{\alpha} \left(\prod_{l=1}^{N_c-1} z_l^{\alpha_l} \right) , \end{aligned} \quad (2.3)$$

where the summation is taken over all roots α , and the notation α_l denotes the number in the l -th position of the root α . For example,

$$[1, 1]_{SU(3)} = 2 + \frac{z_1}{z_2^2} + \frac{1}{z_1 z_2} + \frac{z_1^2}{z_2} + \frac{z_2}{z_1^2} + z_1 z_2 + \frac{z_2^2}{z_1} . \quad (2.4)$$

The group $SO(N_c)$. We note that a special orthogonal group falls into one of the two categories of the classical groups, namely $B_n = SO(2n + 1)$ and $D_n = SO(2n)$. The Lie algebras of B_n and D_n both have the same rank n . The weights of the fundamental (vector) representations of B_n and D_n are respectively $\{0, \pm L_a\}$ and $\{\pm L_a\}$. With this choice, we can write down the characters of the fundamental representations of B_n and D_n respectively as

$$\begin{aligned} [1, 0, \dots, 0]_{B_n}(z_a) &= 1 + \sum_{a=1}^n \left(z_a + \frac{1}{z_a} \right) , \\ [1, 0, \dots, 0]_{D_n}(z_a) &= \sum_{a=1}^n \left(z_a + \frac{1}{z_a} \right) . \end{aligned} \quad (2.5)$$

The adjoint representation of B_n and D_n is the antisymmetric square of the corresponding fundamental representation: $\text{Adj}_{B_n, D_n} = \Lambda^2[1, 0, \dots, 0]_{B_n, D_n}$, and so its character is given by

$$\text{Adj}_{B_n, D_n}(z_a) = \frac{1}{2} \left(([1, 0, \dots, 0]_{B_n, D_n}(z_a))^2 - [1, 0, \dots, 0]_{B_n, D_n}(z_a^2) \right) . \quad (2.6)$$

The roots of the Lie algebras of B_n and D_n are respectively $\{\pm L_a \pm L_b, \pm L_a\}_{a \neq b}$ and $\{\pm L_a \pm L_b\}_{a \neq b}$.

The group $Sp(N_c)$. We shall use the notation such that the rank of the Lie algebra of $Sp(N_c)$ is N_c , the fundamental representation of $Sp(N_c)$ is $2N_c$ dimensional, and $Sp(1)$ is isomorphic to $SU(2)$. We define $L_a = (0, \dots, 0, 1_{a;L}, 0, \dots, 0)$, where the length of the tuple is N_c . The weights of the fundamental representation are $\{\pm L_m\}$. With this choice of L 's, we find the character of the fundamental representation to be

$$[1, 0, \dots, 0]_{Sp(N_c)}(z_a) = \sum_{a=1}^{N_c} \left(z_a + \frac{1}{z_a} \right) . \quad (2.7)$$

The adjoint representation of $Sp(2N_c)$ is the symmetric square of the fundamental representation: $\text{Adj}_{Sp(N_c)} = \text{Sym}^2[1, 0, \dots, 0]_{Sp(N_c)}$, and so its character is given by

$$\text{Adj}_{Sp(N_c)}(z_a) = \frac{1}{2} \left(([1, 0, \dots, 0]_{Sp(N_c)}(z_a))^2 + [1, 0, \dots, 0]_{Sp(N_c)}(z_a^2) \right) . \quad (2.8)$$

The roots of the Lie algebra of $Sp(N_c)$ are $\{\pm L_a \pm L_b\}$, where $1 \leq a, b \leq N_c$.

The group G_2 . The Lie algebra of G_2 has rank 2. We define $L_1 = (2, -1)$, $L_2 = (-1, 1)$. The weights of the fundamental representation are $\{0, \pm L_1, \pm L_2, \pm(L_1 + L_2)\}$. With this choice of weights, the character of the fundamental representation is

$$[1, 0]_{G_2} = 1 + \frac{1}{z_1} + z_1 + \frac{z_1}{z_2} + \frac{z_1^2}{z_2} + \frac{z_2}{z_1^2} + \frac{z_2}{z_1} . \quad (2.9)$$

The adjoint representation $[0, 1]_{G_2}$ is given by

$$\begin{aligned} [0, 1] &= \Lambda^2[1, 0]_{G_2} - [1, 0]_{G_2} \\ &= 2 + \frac{1}{z_1} + z_1 + \frac{z_1^3}{z_2^2} + \frac{1}{z_2} + \frac{z_1}{z_2} + \frac{z_1^2}{z_2} + \frac{z_1^3}{z_2} + z_2 + \frac{z_2}{z_1^3} + \frac{z_2}{z_1^2} + \frac{z_2}{z_1} + \frac{z_2^2}{z_1^3} . \end{aligned} \quad (2.10)$$

The plethystic exponential. The chiral GIOs are *symmetric* functions of the fundamental chiral superfields Q_a^i , the antifundamental chiral superfields \tilde{Q}_i^a , and the adjoint chiral superfield ϕ which transform respectively in the fundamental, the antifundamental, and the adjoint representations of the gauge group G . A convenient combinatorial tool which constructs symmetric products of representations is the **plethystic exponential** [1, 2, 3, 4, 5, 6, 7, 8, 9, 10], which is a *generator for symmetrisation*. To briefly remind the reader, we define the plethystic exponential of a multi-variable function $g(t_1, \dots, t_n)$ that vanishes at the origin, $g(0, \dots, 0) = 0$, to be

$$\text{PE}[g(t_1, \dots, t_n)] := \exp \left(\sum_{r=1}^{\infty} \frac{g(t_1^r, \dots, t_n^r)}{r} \right) . \quad (2.11)$$

Using formula (2.11) and the expansion $-\log(1-x) = \sum_{k=1}^{\infty} x^k/k$, we have, for example,

$$\begin{aligned}
\text{PE} \left[[1, 0, \dots, 0]_{SU(N_c)} \sum_{i=1}^{N_f} t_i \right] &= \frac{1}{\prod_{i=1}^{N_f} \left[(1 - t_i z_1) \left(1 - \frac{t_i}{z_{N_c-1}} \right) \prod_{k=2}^{N_c-1} (1 - t_i \frac{z_k}{z_{k-1}}) \right]} , \\
\text{PE} \left[[1, 0, \dots, 0]_{B_n} \sum_{i=1}^{N_f} t_i \right] &= \frac{1}{\prod_{i=1}^{N_f} \prod_{a=1}^n (1 - t_i)(1 - t_i z_a) \left(1 - \frac{t_i}{z_a} \right)} , \\
\text{PE} \left[[1, 0, \dots, 0]_{D_n} \sum_{i=1}^{N_f} t_i \right] &= \frac{1}{\prod_{i=1}^{N_f} \prod_{a=1}^n (1 - t_i z_a) \left(1 - \frac{t_i}{z_a} \right)} . \tag{2.12}
\end{aligned}$$

The Molien–Weyl formula. We emphasize that, in order to obtain the generating function that counts *gauge invariant* quantities, we need to project the representations of the gauge group generated by the plethystic exponential onto the trivial subrepresentation, which consists of the quantities *invariant* under the action of the gauge group. Using knowledge from representation theory, this can be done by integrating over the whole group. Hence, the generating function for the gauge group G with N_f chiral multiplets in the fundamental representation, N_f chiral multiplets in the antifundamental representation, and 1 chiral multiplet in the adjoint representation is given by

$$g^{(N_f, G)} = \int_G d\mu_G \text{PE} \left[\chi_G^{\text{fund}}(z_a) \sum_{i=1}^{N_f} t_i + \chi_G^{\text{antifund}}(z_a) \sum_{i=1}^{N_f} \tilde{t}_i + \chi_G^{\text{adjoint}}(z^a) s \right] , \tag{2.13}$$

where the notation χ signifies the character. This formula is called the **Molien–Weyl formula** [1, 2, 11, 12, 13, 14, 15]. In the following section, we shall demonstrate in details how to use this formula to count GIOs. We note that the Haar measure for the gauge group μ_G is given by [26]

$$\int_G d\mu_G = \frac{1}{(2\pi i)^r} \oint_{|z_1|=1} \dots \oint_{|z_r|=1} \frac{dz_1}{z_1} \dots \frac{dz_r}{z_r} \prod_{\alpha^+} \left(1 - \prod_{l=1}^r z_l^{\alpha_l^+} \right) , \tag{2.14}$$

where α^+ are positive roots³ of the Lie algebra of the gauge group G and $r = \text{rank } G$.

³We note that the Haar measure we use here is different from those in [1, 2]. The former involves only positive roots and therefore has no Weyl group normalisation. This proves to extensively reduce the amount of computations.

For example,

$$\begin{aligned}
\int_{SU(2)} d\mu_{SU(2)} &= \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} (1 - z^2) , \\
\int_{SU(3)} d\mu_{SU(3)} &= \frac{1}{(2\pi i)^2} \oint_{|z_1|=1} \frac{dz_1}{z_1} \oint_{|z_2|=1} \frac{dz_2}{z_2} (1 - z_1 z_2) \left(1 - \frac{z_1^2}{z_2}\right) \left(1 - \frac{z_2^2}{z_1}\right) , \\
\int_{SO(3)} d\mu_{SO(3)} &= \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} (1 - z) , \\
\int_{SO(4)} d\mu_{SO(4)} &= \frac{1}{(2\pi i)^2} \oint_{|z_1|=1} \frac{dz_1}{z_1} \oint_{|z_2|=1} \frac{dz_2}{z_2} \left(1 - \frac{z_1}{z_2}\right) (1 - z_1 z_2) , \\
\int_{SO(5)} d\mu_{SO(5)} &= \frac{1}{(2\pi i)^2} \oint_{|z_1|=1} \frac{dz_1}{z_1} \oint_{|z_2|=1} \frac{dz_2}{z_2} (1 - z_1)(1 - z_2) \left(1 - \frac{z_1}{z_2}\right) (1 - z_1 z_2) , \\
\int_{Sp(2)} d\mu_{Sp(2)} &= \frac{1}{(2\pi i)^2} \oint_{|z_1|=1} \frac{dz_1}{z_1} \oint_{|z_2|=1} \frac{dz_2}{z_2} (1 - z_1^2)(1 - z_2)(1 - \frac{z_1^2}{z_2})(1 - \frac{z_2^2}{z_1^2}) , \\
\int_{G_2} d\mu_{G_2} &= \frac{1}{(2\pi i)^2} \oint_{|z_1|=1} \frac{dz_1}{z_1} \oint_{|z_2|=1} \frac{dz_2}{z_2} (1 - z_1) \left(1 - \frac{z_1^2}{z_2}\right) \left(1 - \frac{z_1^3}{z_2}\right) (1 - z_2) \times \\
&\quad \left(1 - \frac{z_2}{z_1}\right) \left(1 - \frac{z_2^2}{z_1^3}\right) . \tag{2.15}
\end{aligned}$$

The Hilbert series. A Hilbert series is a function that counts chiral gauge invariant operators and has the interpretation of a partition function at zero temperature and non-zero chemical potentials for global conserved $U(1)$ charges. It can be written as a rational function whose numerator is a polynomial with integer coefficients. Importantly, the powers of the denominators are such that *the leading pole captures the dimension of the manifold*.

The plethystic logarithm. Information about the generators of the moduli space and the relations they satisfy can be computed by using the **plethystic logarithm** [1, 2, 3, 5, 6, 9, 16], which is the inverse function of the plethystic exponential. Using the Möbius function $\mu(r)$ we define:

$$\text{PL}[g(t_1, \dots, t_n)] := \sum_{r=1}^{\infty} \frac{\mu(r) \log g(t_1^r, \dots, t_n^r)}{r} . \tag{2.16}$$

The significance of the series expansion of the plethystic logarithm is that *the first terms with plus sign give the basic generators while the first terms with the minus sign give the constraints between these basic generators*. If the formula (2.16) is an infinite series of terms with plus and minus signs, then the moduli space is not a complete intersection⁴

⁴Mathematicians refer to the dimension of the moduli space (or the order of the pole of the series for $t = 1$) as the **Krull dimension**.

and the constraints in the chiral ring are not trivially generated by relations between the basic generators, but receive stepwise corrections at higher degree. These are the so-called higher syzygies.

3. Dimension of the Moduli Space

At a generic point of the moduli space, the gauge symmetry G is broken completely, and hence there are $d(G)$ broken generators. In the Higgs mechanism, a massless vector multiplet ‘eats’ an entire chiral multiplet to form a massive vector multiplet. Originally, we have $d(G)$ degrees of freedom coming from the chiral superfields in the adjoint representation (which is $d(G)$ dimensional), and $N_\chi d(\square)$ degrees of freedom coming from the N_χ chiral superfields in the fundamental (and antifundamental) representation (which is $d(\square)$ dimensional). Therefore, of the original $d(G) + N_\chi d(\square)$ chiral degrees of freedom, only $[d(G) + N_\chi d(\square)] - d(G) = N_\chi d(\square)$ singlets are left massless. Therefore, the dimension of the moduli space \mathcal{M} is

$$\dim \mathcal{M} = N_\chi d(\square) . \quad (3.1)$$

For the $SU(N_c)$ adjoint SQCD with N_f chiral superfields in the fundamental representation and N_f chiral superfields in the antifundamental representation, we have $N_\chi = 2N_f$ and $d(\square) = N_c$. Therefore,

$$\dim \mathcal{M}_{(N_f, SU(N_c))} = 2N_f N_c . \quad (3.2)$$

For the $Sp(N_c)$ adjoint SQCD with $2N_f$ chiral superfields in the fundamental representation, we have $N_\chi = 2N_f$ and $d(\square) = 2N_c$. Therefore,

$$\dim \mathcal{M}_{(N_f, Sp(N_c))} = 4N_f N_c . \quad (3.3)$$

For the $SO(N_c)$ adjoint SQCD with N_f chiral superfields in the fundamental (vector) representation, we have $N_\chi = N_f$ and $d(\square) = N_c$. Therefore,

$$\dim \mathcal{M}_{(N_f, SO(N_c))} = N_f N_c . \quad (3.4)$$

For the G_2 adjoint SQCD with N_f chiral superfields in the fundamental representation, we have $N_\chi = N_f$ and $d(\square) = 7$. Therefore,

$$\dim \mathcal{M}_{(N_f, G_2)} = 7N_f . \quad (3.5)$$

4. The $SU(N_c)$ Gauge Groups

Let us consider the $SU(N_c)$ theory with N_f chiral superfields transforming in the fundamental representation, N_f chiral superfields transforming in the antifundamental representation (*i.e.* N_f flavours), and 1 chiral superfield transforming in the adjoint representation. The anomaly-free global symmetry of this theory [19] is $SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_{R_1} \times U(1)_{R_2}$.

4.1 Examples of Hilbert Series

Below we shall derive Hilbert series for various cases.

4.1.1 The $SU(2)$ Gauge Group

We start the analysis by the simplest case of the $SU(2)$ gauge theory with $2N_f$ chiral superfields transforming in the fundamental representation (N_f flavours)⁵ and 1 chiral multiplet transforming in the adjoint representation. The Molien–Weyl formula can be written explicitly as:

$$\begin{aligned} g^{(N_f, SU(2))}(s, t) &= \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} (1 - z^2) \text{PE} [2N_f[1]t + [2]s] \\ &= \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} \frac{1 - z^2}{(1 - tz)^{2N_f} (1 - \frac{t}{z})^{2N_f} (1 - s)(1 - sz^2)(1 - \frac{s}{z^2})} . \end{aligned} \quad (4.1)$$

Noting that $0 < |t|, |s| < 1$, we use the residue theorem with the poles $z = t, \sqrt{s}, -\sqrt{s}$

⁵Note that the number of fundamental chiral superfields must be even due to the global \mathbb{Z}_2 anomaly.

and find that

$$\begin{aligned}
g^{(1,SU(2))}(s,t) &= \frac{1+st^2}{(1-s^2)(1-t^2)(1-st^2)^2} \\
&= 1 + s^2 + s^4 + t^2 + 3st^2 + s^2t^2 + 3s^3t^2 + s^4t^2 + 3s^5t^2 + t^4 + 3st^4 + 6s^2t^4 + \\
&\quad 3s^3t^4 + 6s^4t^4 + 3s^5t^4 + O(s^6)O(t^6) , \\
g^{(2,SU(2))}(s,t) &= \frac{1+t^2+6st^2-9st^4+s^2t^4+st^6-9s^2t^6+6s^2t^8+s^3t^8+s^3t^{10}}{(1-s^2)(1-t^2)^5(1-st^2)^4} \\
&= 1 + s^2 + s^4 + 6t^2 + 10st^2 + 6s^2t^2 + 10s^3t^2 + 6s^4t^2 + 10s^5t^2 + 20t^4 + \\
&\quad 45st^4 + 55s^2t^4 + 45s^3t^4 + 55s^4t^4 + 45s^5t^4 + O(s^6)O(t^6) , \\
g^{(3,SU(2))}(s,t) &= 1 + s^2 + s^4 + 15t^2 + 21st^2 + 15s^2t^2 + 21s^3t^2 + 15s^4t^2 + 21s^5t^2 + 105t^4 + \\
&\quad 210st^4 + 231s^2t^4 + 210s^3t^4 + 231s^4t^4 + 210s^5t^4 + O(s^6)O(t^6) , \\
g^{(4,SU(2))}(s,t) &= 1 + s^2 + s^4 + 28t^2 + 36st^2 + 28s^2t^2 + 36s^3t^2 + 28s^4t^2 + 36s^5t^2 + 336t^4 + \\
&\quad 630st^4 + 666s^2t^4 + 630s^3t^4 + 666s^4t^4 + 630s^5t^4 + O(s^6)O(t^6) , \\
g^{(5,SU(2))}(s,t) &= 1 + s^2 + s^4 + 45t^2 + 55st^2 + 45s^2t^2 + 55s^3t^2 + 45s^4t^2 + 55s^5t^2 + 825t^4 + \\
&\quad 1485st^4 + 1540s^2t^4 + 1485s^3t^4 + 1540s^4t^4 + 1485s^5t^4 + O(s^6)O(t^6) . \quad (4.2)
\end{aligned}$$

Looking at these generating functions, it is possible to predict the order of the numerator and the terms in the denominator of the generating function for a case with N_f fundamental quarks:

$$g^{(N_f,SU(2))} = \frac{P_{(2N_f-1),(8N_f-6)}(s,t)}{(1-s^2)(1-t^2)^{4N_f-3}(1-st^2)^{2N_f}}, \quad (4.3)$$

where $P_{a,b}(s,t)$ is a polynomial of degree a in s and of degree b in t . The calculation is simpler when we make a further identification $s = t$. In which case, we can write down a general form of the generating function:

$$g^{(N_f,SU(2))}(t) = \frac{P_{8N_f-6}(t)}{(1+t)^{2N_f-3}(1-t^2)^{2N_f}(1-t^3)^{2N_f}}, \quad (4.4)$$

where $P_{8N_f-6}(t)$ is a palindromic polynomial of degree $8N_f - 6$ in t with $P_{8N_f-6}(1) \neq 0$ for all N_f . Observe that the order of the pole at $t = 1$ of $g^{(N_f,SU(2))}(t)$ is $4N_f$. Therefore, the dimension of the moduli space $\mathcal{M}_{(N_f,SU(2))}$ is $4N_f$, in agreement with (3.2) and (3.3).

Character expansion. We can write down the generating function for an *arbitrary* number of flavours N_f in terms of representations of the global symmetry $SU(2N_f)$ as

follows:

$$\begin{aligned}
g^{(N_f, SU(2))} &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{m=0}^{\infty} [2n_1, n_2, 0, \dots, 0] s^{n_1+2m} t^{2n_1+2n_2} \\
&= \frac{1}{1-s^2} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} [2n_1, n_2, 0, \dots, 0] s^{n_1} t^{2n_1+2n_2} .
\end{aligned} \tag{4.5}$$

We emphasise that $\frac{1}{1-s^2}$ does factor out from the character expansion.

Plethystic logarithms. We shall calculate the plethystic logarithms of the generating functions in (4.2) using formula (2.16):

$$\begin{aligned}
\text{PL}[g^{(1, SU(2))}(s, t)] &= s^2 + t^2 + 3st^2 - s^2t^4 , \\
\text{PL}[g^{(2, SU(2))}(s, t)] &= s^2 + 6t^2 + 10st^2 - t^4 - 15st^4 - 20s^2t^4 + O(s^6)O(t^6) , \\
\text{PL}[g^{(3, SU(2))}(s, t)] &= s^2 + 15t^2 + 21st^2 - 15t^4 - 105st^4 - 105s^2t^4 + O(s^6)O(t^6) \\
\text{PL}[g^{(4, SU(2))}(s, t)] &= s^2 + 28t^2 + 36st^2 - 70t^4 - 378st^4 - 336s^2t^4 + O(s^6)O(t^6) \\
\text{PL}[g^{(5, SU(2))}(s, t)] &= s^2 + 45t^2 + 55st^2 - 210t^4 - 990st^4 - 825s^2t^4 + O(s^6)O(t^6) .
\end{aligned} \tag{4.6}$$

Observe that only plethystic logarithm of $g^{(1, SU(2))}$ is a polynomial. Therefore, the moduli space of the $SU(2)$ gauge theory with 1 flavour and 1 adjoint matter is a *complete intersection*.

Generators of the GIOs. According to (4.6), we see that there are only 3 types of generators of the GIOs in the $SU(2)$ theory, namely

$$\begin{aligned}
\text{Casimir invariants : } s^2 &\rightarrow u \equiv \text{Tr}(\phi^2) & : [0, \dots, 0] & \text{1 dimensional} , \\
\text{Mesons : } t^2 &\rightarrow M^{ij} \equiv \epsilon^{ab} Q_a^i Q_b^j & : [0, 1, \dots, 0] & \binom{2N_f}{2} \text{ dimensional} , \\
\text{Adjoint mesons : } st^2 &\rightarrow A^{ij} \equiv \epsilon^{ab} \epsilon^{cd} Q_a^i \phi_{bc} Q_d^j & : [2, 0, \dots, 0] & N_f(2N_f + 1) \text{ dimensional} .
\end{aligned}$$

Note that the total number of generators is quadratic in N_f ,

$$1 + \binom{2N_f}{2} + N_f(2N_f + 1) = 4N_f^2 + 1 . \tag{4.7}$$

Relations between the generators. From plethystic logarithms (4.6), we see that there are 3 types of basic relations:

- **Order t^4 :** The relations are known from the theory without adjoint:

$$\text{Pf } M = \epsilon_{i_1 \dots i_{2N_f}} M^{i_1 i_2} M^{i_3 i_4} = 0 . \tag{4.8}$$

They transform in the $SU(2N_f)$ representation $[0, 0, 0, 1, 0, \dots, 0]$. We note that this is contained in the decomposition of the symmetric square of the representation $[0, 1, 0, \dots, 0]$ at order t^2 .

- **Order st^4 :** The relations transform in the representation $[1, 0, 1, 0, \dots, 0]$, which is contained in the decomposition of the antisymmetric product of the representation $[2, 0, \dots, 0]$ at order st^2 and the representation $[0, 1, 0, \dots, 0]$ at order t^2 .
- **Order s^2t^4 :** The relations transform in the representation $[0, 2, 0, \dots, 0]$, which is contained in the decomposition of the symmetric square of the representation $[2, 0, \dots, 0]$ at order st^2 . In the case of 1 flavour, there is only 1 basic relation which can be written out explicitly as

$$A^{11}A^{22} - (A^{12})^2 + \frac{1}{2}u(M^{12})^2 = 0. \quad (4.9)$$

In summary, for the $SU(2)$ theory, we have the basic relations which transform in the $SU(2N_f)$ representations $[0, 0, 0, 1, 0, \dots, 0]$, $[1, 0, 1, 0, \dots, 0]$, and $[0, 2, 0, \dots, 0]$.

Therefore, we may write down a general expression of the plethystic logarithm in terms of $SU(2N_f)$ representations as

$$\begin{aligned} \text{PL}[g^{(N_f, SU(2))}(s, t)] &= [0, \dots, 0]s^2 + [0, 1, \dots, 0]t^2 + [2, 0, \dots, 0]st^2 \\ &\quad - [0, 0, 0, 1, 0, \dots, 0]t^4 - [1, 0, 1, 0, \dots, 0]st^4 - [0, 2, 0, \dots, 0]s^2t^4 + O(s^6)O(t^6). \end{aligned} \quad (4.10)$$

4.1.2 The $SU(3)$ Gauge Group

Now let us turn to the $SU(3)$ theory with N_f flavours and 1 adjoint matter. We have N_f chiral superfields transforming in the fundamental representation, N_f chiral superfields transforming in the antifundamental representation, and 1 chiral superfield transforming in the adjoint representation. Therefore, we can apply the Molien–Weyl formula to our theory as follows:

$$\begin{aligned} g^{(N_f, SU(3))} &= \frac{1}{(2\pi i)^2} \oint_{|z_1|=1} \frac{dz_1}{z_1} \oint_{|z_2|=1} \frac{dz_2}{z_2} (1 - z_1 z_2) \left(1 - \frac{z_1^2}{z_2}\right) \left(1 - \frac{z_2^2}{z_1}\right) \text{PE} [N_f[1, 0]t + N_f[0, 1]\tilde{t} + [1, 1]s] \\ &= \frac{1}{(2\pi i)^2} \oint_{|z_1|=1} \frac{dz_1}{z_1} \oint_{|z_2|=1} \frac{dz_2}{z_2} \frac{(1 - z_1 z_2) \left(1 - \frac{z_1^2}{z_2}\right) \left(1 - \frac{z_2^2}{z_1}\right)}{\left((1 - tz_1)(1 - t\frac{z_2}{z_1})(1 - \frac{t}{z_2})\right)^{N_f} \left((1 - \tilde{t}z_2)(1 - \tilde{t}\frac{z_1}{z_2})(1 - \frac{\tilde{t}}{z_1})\right)^{N_f}} \times \\ &\quad \frac{1}{(1 - sz_1 z_2)(1 - s\frac{z_1^2}{z_2})(1 - s\frac{z_2^2}{z_1})(1 - s\frac{z_1}{z_2})(1 - s\frac{z_2}{z_1})(1 - s\frac{1}{z_1 z_2})(1 - s)^2}. \end{aligned} \quad (4.11)$$

Applying the residue theorem, we have

$$\begin{aligned}
g^{(1, SU(3))}(s, t, \tilde{t}) &= \frac{1 - s^6 t^3 \tilde{t}^3}{(1 - s^2)(1 - s^3)(1 - t\tilde{t})(1 - st\tilde{t})(1 - s^2 t\tilde{t})(1 - s^3 t^3)(1 - s^3 \tilde{t}^3)} \\
&= 1 + s^2 + s^3 + s^4 + s^5 + 2s^6 + s^3 t^3 + s^5 t^3 + s^6 t^3 + t\tilde{t} + st\tilde{t} + 2s^2 t\tilde{t} + \\
&\quad 2s^3 t\tilde{t} + 3s^4 t\tilde{t} + 3s^5 t\tilde{t} + 4s^6 t\tilde{t} + t^2 \tilde{t}^2 + st^2 \tilde{t}^2 + 3s^2 t^2 \tilde{t}^2 + 3s^3 t^2 \tilde{t}^2 + \\
&\quad 5s^4 t^2 \tilde{t}^2 + 5s^5 t^2 \tilde{t}^2 + 7s^6 t^2 \tilde{t}^2 + s^3 \tilde{t}^3 + s^5 \tilde{t}^3 + s^6 \tilde{t}^3 + t^3 \tilde{t}^3 + st^3 \tilde{t}^3 + \\
&\quad 3s^2 t^3 \tilde{t}^3 + 4s^3 t^3 \tilde{t}^3 + 6s^4 t^3 \tilde{t}^3 + 7s^5 t^3 \tilde{t}^3 + 10s^6 t^3 \tilde{t}^3 + O(s^7)O(t^4)O(\tilde{t}^4), \\
g^{(2, SU(3))}(s, t, \tilde{t}) &= 1 + s^2 + s^3 + s^4 + s^5 + 2s^6 + 2st^3 + 2s^2 t^3 + 6s^3 t^3 + 4s^4 t^3 + 8s^5 t^3 + 8s^6 t^3 + \\
&\quad 4t\tilde{t} + 4st\tilde{t} + 8s^2 t\tilde{t} + 8s^3 t\tilde{t} + 12s^4 t\tilde{t} + 12s^5 t\tilde{t} + 16s^6 t\tilde{t} + 10t^2 \tilde{t}^2 + 16st^2 \tilde{t}^2 + \\
&\quad 35s^2 t^2 \tilde{t}^2 + 41s^3 t^2 \tilde{t}^2 + 60s^4 t^2 \tilde{t}^2 + 66s^5 t^2 \tilde{t}^2 + 85s^6 t^2 \tilde{t}^2 + 2s\tilde{t}^3 + 2s^2 \tilde{t}^3 + \\
&\quad 6s^3 \tilde{t}^3 + 4s^4 \tilde{t}^3 + 8s^5 \tilde{t}^3 + 8s^6 \tilde{t}^3 + 20t^3 \tilde{t}^3 + 40st^3 \tilde{t}^3 + 96s^2 t^3 \tilde{t}^3 + 136s^3 t^3 \tilde{t}^3 + \\
&\quad 204s^4 t^3 \tilde{t}^3 + 244s^5 t^3 \tilde{t}^3 + 316s^6 t^3 \tilde{t}^3 + O(s^7)O(t^4)O(\tilde{t}^4), \\
g^{(3, SU(3))}(s, t, \tilde{t}) &= 1 + s^2 + s^3 + s^4 + s^5 + 2s^6 + t^3 + 8st^3 + 9s^2 t^3 + 19s^3 t^3 + 17s^4 t^3 + 27s^5 t^3 + \\
&\quad 28s^6 t^3 + 9t\tilde{t} + 9st\tilde{t} + 18s^2 t\tilde{t} + 18s^3 t\tilde{t} + 27s^4 t\tilde{t} + 27s^5 t\tilde{t} + 36s^6 t\tilde{t} + \\
&\quad 45t^2 \tilde{t}^2 + 81st^2 \tilde{t}^2 + 162s^2 t^2 \tilde{t}^2 + 198s^3 t^2 \tilde{t}^2 + 279s^4 t^2 \tilde{t}^2 + 315s^5 t^2 \tilde{t}^2 + \\
&\quad 396s^6 t^2 \tilde{t}^2 + \tilde{t}^3 + 8s\tilde{t}^3 + 9s^2 \tilde{t}^3 + 19s^3 \tilde{t}^3 + 17s^4 \tilde{t}^3 + 27s^5 \tilde{t}^3 + 28s^6 \tilde{t}^3 + \\
&\quad 165t^3 \tilde{t}^3 + 404st^3 \tilde{t}^3 + 893s^2 t^3 \tilde{t}^3 + 1301s^3 t^3 \tilde{t}^3 + 1881s^4 t^3 \tilde{t}^3 + 2289s^5 t^3 \tilde{t}^3 + \\
&\quad 2878s^6 t^3 \tilde{t}^3 + O(s^7)O(t^4)O(\tilde{t}^4), \\
g^{(4, SU(3))}(s, t, \tilde{t}) &= 1 + s^2 + s^3 + s^4 + s^5 + 2s^6 + 4t^3 + 20st^3 + 24s^2 t^3 + 44s^3 t^3 + 44s^4 t^3 + 64s^5 t^3 + \\
&\quad 68s^6 t^3 + 16t\tilde{t} + 16st\tilde{t} + 32s^2 t\tilde{t} + 32s^3 t\tilde{t} + 48s^4 t\tilde{t} + 48s^5 t\tilde{t} + 64s^6 t\tilde{t} + \\
&\quad 136t^2 \tilde{t}^2 + 256st^2 \tilde{t}^2 + 492s^2 t^2 \tilde{t}^2 + 612s^3 t^2 \tilde{t}^2 + 848s^4 t^2 \tilde{t}^2 + 968s^5 t^2 \tilde{t}^2 + \\
&\quad 1204s^6 t^2 \tilde{t}^2 + 4\tilde{t}^3 + 20s\tilde{t}^3 + 24s^2 \tilde{t}^3 + 44s^3 \tilde{t}^3 + 44s^4 \tilde{t}^3 + 64s^5 \tilde{t}^3 + 68s^6 \tilde{t}^3 + \\
&\quad 816t^3 \tilde{t}^3 + 2160st^3 \tilde{t}^3 + 4576s^2 t^3 \tilde{t}^3 + 6736s^3 t^3 \tilde{t}^3 + 9536s^4 t^3 \tilde{t}^3 + 11696s^5 t^3 \tilde{t}^3 + \\
&\quad 14512s^6 t^3 \tilde{t}^3 + O(s^7)O(t^4)O(\tilde{t}^4), \\
g^{(5, SU(3))}(s, t, \tilde{t}) &= 1 + s^2 + s^3 + s^4 + s^5 + 2s^6 + 10t^3 + 40st^3 + 50s^2 t^3 + 85s^3 t^3 + 90s^4 t^3 + 125s^5 t^3 + \\
&\quad 135s^6 t^3 + 25t\tilde{t} + 25st\tilde{t} + 50s^2 t\tilde{t} + 50s^3 t\tilde{t} + 75s^4 t\tilde{t} + 75s^5 t\tilde{t} + 100s^6 t\tilde{t} + \\
&\quad 325t^2 \tilde{t}^2 + 625st^2 \tilde{t}^2 + 1175s^2 t^2 \tilde{t}^2 + 1475s^3 t^2 \tilde{t}^2 + 2025s^4 t^2 \tilde{t}^2 + 2325s^5 t^2 \tilde{t}^2 + \\
&\quad 2875s^6 t^2 \tilde{t}^2 + 10\tilde{t}^3 + 40s\tilde{t}^3 + 50s^2 \tilde{t}^3 + 85s^3 \tilde{t}^3 + 90s^4 \tilde{t}^3 + 125s^5 \tilde{t}^3 + 135s^6 \tilde{t}^3 + \\
&\quad 2925t^3 \tilde{t}^3 + 8025st^3 \tilde{t}^3 + 16575s^2 t^3 \tilde{t}^3 + 24500s^3 t^3 \tilde{t}^3 + 34250s^4 t^3 \tilde{t}^3 + 42175s^5 t^3 \tilde{t}^3 + \\
&\quad 51950s^6 t^3 \tilde{t}^3 + O(s^7)O(t^4)O(\tilde{t}^4), \\
g^{(6, SU(3))}(s, t, \tilde{t}) &= 1 + s^2 + s^3 + s^4 + s^5 + 2s^6 + 20t^3 + 70st^3 + 90s^2 t^3 + 146s^3 t^3 + 160s^4 t^3 + 216s^5 t^3 + \\
&\quad 236s^6 t^3 + 36t\tilde{t} + 36st\tilde{t} + 72s^2 t\tilde{t} + 72s^3 t\tilde{t} + 108s^4 t\tilde{t} + 108s^5 t\tilde{t} + 144s^6 t\tilde{t} + 666t^2 \tilde{t}^2 + \\
&\quad 1296st^2 \tilde{t}^2 + 2403s^2 t^2 \tilde{t}^2 + 3033s^3 t^2 \tilde{t}^2 + 4140s^4 t^2 \tilde{t}^2 + 4770s^5 t^2 \tilde{t}^2 + 5877s^6 t^2 \tilde{t}^2 + \\
&\quad 20\tilde{t}^3 + 70s\tilde{t}^3 + 90s^2 \tilde{t}^3 + 146s^3 \tilde{t}^3 + 160s^4 \tilde{t}^3 + 216s^5 \tilde{t}^3 + 236s^6 \tilde{t}^3 + 8436t^3 \tilde{t}^3 + \\
&\quad 23576st^3 \tilde{t}^3 + 47888s^2 t^3 \tilde{t}^3 + 70904s^3 t^3 \tilde{t}^3 + 98316s^4 t^3 \tilde{t}^3 + 121332s^5 t^3 \tilde{t}^3 + \\
&\quad 148780s^6 t^3 \tilde{t}^3 + O(s^7)O(t^4)O(\tilde{t}^4). \tag{4.12}
\end{aligned}$$

We remark that, although these results seem to be rather lengthy, they contain information which proves to be extremely useful for analyses of the chiral ring. As we shall see from plethystic logarithms, $s^6 t^3 \tilde{t}^3$ is the minimum order up to which Hilbert series contain all necessary information about the generators and their basic relations.

The calculation is significantly simpler when we make an identification $s = t = \tilde{t}$. In which case, we can write down a general form of the generating function:

$$g^{(N_f, SU(3))}(t) = \frac{P_{14N_f-10}(t)}{(1-t)^{N_f-1}(1-t^2)^{N_f+2}(1-t^3)^{N_f+1}(1-t^4)^{2N_f-2}(1-t^6)^{N_f}}, \quad (4.13)$$

where $P_{14N_f-10}(t)$ is a palindromic polynomial of degree $14N_f - 10$ with $P_{14N_f-10}(1)$ being a non-zero number for any number of flavour. Observe that the order of the pole at $t = 1$ of $g^{(N_f, SU(3))}(t)$ is $6N_f$. Therefore, the dimension of the moduli space $\mathcal{M}_{(N_f, SU(3))}$ is $6N_f$ in agreement with (3.2).

Plethystic logarithms. We shall calculate plethystic logarithms of generating functions using formula (2.16):

$$\begin{aligned} \text{PL}[g^{(1, SU(3))}(s, t, \tilde{t})] &= s^2 + s^3 + t\tilde{t} + st\tilde{t} + s^2 t\tilde{t} + s^3 t^3 + s^3 \tilde{t}^3 - s^6 t^3 \tilde{t}^3, \\ \text{PL}[g^{(2, SU(3))}(s, t, \tilde{t})] &= s^2 + s^3 + 2st^3 + 2s^2 t^3 + 4s^3 t^3 + 4t\tilde{t} + 4st\tilde{t} + 4s^2 t\tilde{t} - s^2 t^2 \tilde{t}^2 - s^3 t^2 \tilde{t}^2 - s^4 t^2 \tilde{t}^2 \\ &\quad + 2s\tilde{t}^3 + 2s^2 \tilde{t}^3 + 4s^3 \tilde{t}^3 - 4s^2 t^3 \tilde{t}^3 - 8s^3 t^3 \tilde{t}^3 - 20s^4 t^3 \tilde{t}^3 - 16s^5 t^3 \tilde{t}^3 - 16s^6 t^3 \tilde{t}^3 \\ &\quad + O(s^7)O(t^4)O(\tilde{t}^4), \\ \text{PL}[g^{(3, SU(3))}(s, t, \tilde{t})] &= s^2 + s^3 + t^3 + 8st^3 + 8s^2 t^3 + 10s^3 t^3 + 9t\tilde{t} + 9st\tilde{t} + 9s^2 t\tilde{t} - 9s^2 t^2 \tilde{t}^2 - 9s^3 t^2 \tilde{t}^2 \\ &\quad - 9s^4 t^2 \tilde{t}^2 + \tilde{t}^3 + 8s\tilde{t}^3 + 8s^2 \tilde{t}^3 + 10s^3 \tilde{t}^3 - t^3 \tilde{t}^3 - 17st^3 \tilde{t}^3 - 81s^2 t^3 \tilde{t}^3 - 148s^3 t^3 \tilde{t}^3 \\ &\quad - 207s^4 t^3 \tilde{t}^3 - 143s^5 t^3 \tilde{t}^3 - 84s^6 t^3 \tilde{t}^3 + O(s^7)O(t^4)O(\tilde{t}^4), \\ \text{PL}[g^{(4, SU(3))}(s, t, \tilde{t})] &= s^2 + s^3 + 4t^3 + 20st^3 + 20s^2 t^3 + 20s^3 t^3 + 16t\tilde{t} + 16st\tilde{t} + 16s^2 t\tilde{t} - 36s^2 t^2 \tilde{t}^2 \\ &\quad - 36s^3 t^2 \tilde{t}^2 - 36s^4 t^2 \tilde{t}^2 + 4\tilde{t}^3 + 20s\tilde{t}^3 + 20s^2 \tilde{t}^3 + 20s^3 \tilde{t}^3 - 16t^3 \tilde{t}^3 - 176st^3 \tilde{t}^3 \\ &\quad - 576s^2 t^3 \tilde{t}^3 - 960s^3 t^3 \tilde{t}^3 - 1024s^4 t^3 \tilde{t}^3 - 624s^5 t^3 \tilde{t}^3 - 240s^6 t^3 \tilde{t}^3 \\ &\quad + O(s^7)O(t^4)O(\tilde{t}^4), \\ \text{PL}[g^{(5, SU(3))}(s, t, \tilde{t})] &= s^2 + s^3 + 10t^3 + 40st^3 + 40s^2 t^3 + 35s^3 t^3 + 25t\tilde{t} + 25st\tilde{t} + 25s^2 t\tilde{t} - 100s^2 t^2 \tilde{t}^2 \\ &\quad - 100s^3 t^2 \tilde{t}^2 - 100s^4 t^2 \tilde{t}^2 + 10\tilde{t}^3 + 40s\tilde{t}^3 + 40s^2 \tilde{t}^3 + 35s^3 \tilde{t}^3 - 100t^3 \tilde{t}^3 - 900st^3 \tilde{t}^3 \\ &\quad - 2500s^2 t^3 \tilde{t}^3 - 3900s^3 t^3 \tilde{t}^3 - 3500s^4 t^3 \tilde{t}^3 - 1900s^5 t^3 \tilde{t}^3 - 425s^6 t^3 \tilde{t}^3 \\ &\quad + O(s^7)O(t^4)O(\tilde{t}^4), \\ \text{PL}[g^{(6, SU(3))}(s, t, \tilde{t})] &= s^2 + s^3 + 20t^3 + 70st^3 + 70s^2 t^3 + 56s^3 t^3 + 36t\tilde{t} + 36st\tilde{t} + 36s^2 t\tilde{t} - 225s^2 t^2 \tilde{t}^2 \\ &\quad - 225s^3 t^2 \tilde{t}^2 - 225s^4 t^2 \tilde{t}^2 + 20\tilde{t}^3 + 70s\tilde{t}^3 + 70s^2 \tilde{t}^3 + 56s^3 \tilde{t}^3 - 400t^3 \tilde{t}^3 - 3200st^3 \tilde{t}^3 \\ &\quad - 8100s^2 t^3 \tilde{t}^3 - 12040s^3 t^3 \tilde{t}^3 - 9540s^4 t^3 \tilde{t}^3 - 4640s^5 t^3 \tilde{t}^3 - 336s^6 t^3 \tilde{t}^3 \\ &\quad + O(s^7)O(t^4)O(\tilde{t}^4). \end{aligned} \quad (4.14)$$

As for the case of $N_c = 2$, the moduli space of the $SU(3)$ theory with 1 flavour and 1 adjoint matter is a complete intersection.

Generators of the GIOs. Armed with plethystic logarithms, we can write down the generators of the GIOs.

$$\begin{aligned}
\text{Casimir invariants : } s^k &\rightarrow u_k = \text{Tr}(\phi^k), \quad k = 2, 3 \\
&\quad [0, \dots, 0; 0, \dots, 0] \quad 1 \text{ dimensional} , \\
\text{Mesons : } t\tilde{t} &\rightarrow M_j^i = Q_a^i \tilde{Q}_j^a \\
&\quad [1, 0, \dots, 0; 0, \dots, 0, 1] \quad N_f^2 \text{ dimensional} , \\
\text{Adjoint mesons : } s^l t\tilde{t} &\rightarrow (A_l)_j^i = Q_a^i (\phi^l)_b^a \tilde{Q}_j^b, \quad l = 1, 2 \\
&\quad [1, 0, \dots, 0; 0, \dots, 0, 1] \quad N_f^2 \text{ dimensional} , \\
\text{Baryons : } t^3 &\rightarrow B^{i_1 i_2 i_3} = \epsilon^{a_1 a_2 a_3} Q_{a_1}^{i_1} Q_{a_2}^{i_2} Q_{a_3}^{i_3} \\
&\quad [0, 0, 1, 0, \dots, 0; 0, \dots, 0] \quad \binom{N_f}{3} \text{ dimensional} , \\
\text{Antibaryons : } \tilde{t}^3 &\rightarrow \tilde{B}_{i_1 i_2 i_3}^a = \epsilon_{a_1 a_2 a_3} \tilde{Q}_{i_1}^{a_1} \tilde{Q}_{i_2}^{a_2} \tilde{Q}_{i_3}^{a_3} \\
&\quad [0, \dots, 0; 0, \dots, 1, 0, 0] \quad \binom{N_f}{3} \text{ dimensional} .
\end{aligned}$$

In addition, we have **adjoint baryons**:

$$\begin{aligned}
st^3 &\rightarrow \mathcal{B}_{0,0,1}^{i_1 i_2 j_1} = \epsilon^{a_1 a_2 b_1} Q_{a_1}^{i_1} Q_{a_2}^{i_2} (P_1)_{b_1}^{j_1} \\
&\quad [1, 1, 0, \dots, 0; 0, \dots, 0]^* \quad \frac{1}{3}(N_f - 1)N_f(N_f + 1) \text{ dimensional} , \\
s^2 t^3 &\rightarrow \mathcal{B}_{0,1,1}^{i_1 j_1 j_2} = \epsilon^{a_1 b_1 b_2} Q_{a_1}^{i_1} (P_1)_{b_1}^{j_1} (P_1)_{b_2}^{j_2}, \quad \mathcal{B}_{0,0,2}^{i_1 i_2 j_1} = \epsilon^{a_1 a_2 b_1} Q_{a_1}^{i_1} Q_{a_2}^{i_2} (P_2)_{b_1}^{j_1} \\
&\quad [1, 1, 0, \dots, 0; 0, \dots, 0]^{**} \quad \frac{1}{3}(N_f - 1)N_f(N_f + 1) \text{ dimensional} , \\
s^3 t^3 &\rightarrow \mathcal{B}_{1,1,1}^{ijk} = \epsilon^{abc} (P_1)_a^i (P_1)_b^j (P_1)_c^k, \quad \mathcal{B}_{0,1,2}^{i_1 j_1 k_1} = \epsilon^{a_1 b_1 c_1} Q_{a_1}^{i_1} (P_1)_{b_1}^{j_1} (P_2)_{c_2}^{k_2}, \quad \mathcal{B}_{0,0,3}^{i_1 i_2 j_1} = \epsilon^{a_1 a_2 b_1} Q_{a_1}^{i_1} Q_{a_2}^{i_2} (P_3)_{b_1}^{j_1} \\
&\quad [3, 0, \dots, 0; 0, \dots, 0]^{***} \quad \frac{1}{3!}N_f(N_f + 1)(N_f + 2) \text{ dimensional} ,
\end{aligned}$$

where $(P_m)_a^i = \phi_a^{b_1} \phi_{b_1}^{b_2} \dots \phi_{b_{m-1}}^{b_m} Q_{b_m}^i$, and the subscript of \mathcal{B} indicates the partition of the power of s in the adjoint baryon. Moreover, in the same spirit as antibaryons, we also have **adjoint antibaryons** which transform in the conjugate representations of adjoint baryons.

***The generator at order st^3 .** Note that the generator $\mathcal{B}_{0,0,1}^{i_1 i_2 j_1}$ is subject to a relation:

$$\mathcal{B}_{0,0,1}^{[i_1 i_2 j_1]} = 0 , \tag{4.15}$$

where the square bracket denotes an antisymmetrisation without a normalisation factor. This means that the completely antisymmetric part, which transforms in the $SU(N_f)$ representation $[0, 0, 1, 0, \dots, 0]$, vanishes. Note that we can construct the generator by considering the following $SU(N_f)$ tensor product:

$$[0, 1, 0, \dots, 0] \times [1, 0, \dots, 0] = [1, 1, 0, \dots, 0] + [0, 0, 1, 0, \dots, 0] .$$

Therefore, after taking (4.15) into account, we conclude that $\mathcal{B}_{0,0,1}^{i_1 i_2 j_1}$ transforms in the $SU(N_f) \times SU(N_f)$ representation $[1, 1, 0, \dots, 0; 0, \dots, 0]$, as stated in the list above.

****Two generators at order s^2t^3 .** We can construct *each* of the generators $\mathcal{B}_{0,1,1}$ and $\mathcal{B}_{0,0,2}$ by considering the following $SU(N_f)$ tensor product:

$$[0, 1, 0, \dots, 0] \times [1, 0, \dots, 0] = [1, 1, 0, \dots, 0] + [0, 0, 1, 0, \dots, 0] .$$

Therefore, if there were no relations, we would say that the generators transform in $2[1, 1, 0, \dots, 0] + 2[0, 0, 1, 0, \dots, 0]$. However, $\mathcal{B}_{0,1,1}$ and $\mathcal{B}_{0,0,2}$ are subject to the relations:

$$\mathcal{B}_{0,0,2}^{ijk} = -\mathcal{B}_{0,1,1}^{[ij]k} , \quad \mathcal{B}_{0,0,2}^{[ijk]} = -2\mathcal{B}_{0,1,1}^{[ijk]} , \quad (4.16)$$

which transforms respectively in the $SU(N_f)$ representation $[1, 1, 0, \dots, 0] + [0, 0, 1, 0, \dots, 0]$ and $[0, 0, 1, 0, \dots, 0]$. Therefore, we are left with the global $SU(N_f) \times SU(N_f)$ representation $[1, 1, 0, \dots, 0; 0, \dots, 0]$, as stated in the above list.

*****Three generators at order s^3t^3 .** We can construct the generators $\mathcal{B}_{1,1,1}$, $\mathcal{B}_{0,1,2}$, $\mathcal{B}_{0,0,3}$ from the following $SU(N_f)$ tensor products:

$$\begin{aligned} \Lambda^3[1, 0, \dots, 0] &= [0, 0, 1, 0, \dots, 0] , \\ [1, 0, \dots, 0]^3 &= [3, 0, \dots, 0] + 2[1, 1, 0, \dots, 0] + [0, 0, 1, 0, \dots, 0] , \\ [0, 1, 0, \dots, 0] \times [1, 0, \dots, 0] &= [1, 1, 0, \dots, 0] + [0, 0, 1, 0, \dots, 0] . \end{aligned}$$

Therefore, if there were no relations, we would say that the generators transform in

$$[3, 0, \dots, 0] + 3[1, 1, 0, \dots, 0] + 3[0, 0, 1, 0, \dots, 0] .$$

However, these generators are subject to the relations:

$$\mathcal{B}_{0,0,3}^{ijk} = -\mathcal{B}_{0,1,2}^{[ij]k} , \quad (4.17)$$

$$\mathcal{B}_{1,1,1}^{ijk} = -\mathcal{B}_{0,1,2}^{i[jk]} , \quad (4.18)$$

$$2\mathcal{B}_{1,1,1}^{ijk} + \mathcal{B}_{0,0,3}^{ikj} = -\mathcal{B}_{0,1,2}^{[i[jk]]} \equiv -\left(\mathcal{B}_{0,1,2}^{ijk} - \mathcal{B}_{0,1,2}^{kji}\right) . \quad (4.19)$$

These relations transform in the $SU(N_f)$ representation $3[1, 1, 0, \dots, 0] + 3[0, 0, 1, 0, \dots, 0]$. Therefore, we are left with the global $SU(N_f) \times SU(N_f)$ representation $[3, 0, \dots, 0; 0, \dots, 0]$, as stated in the above list.

Total number of generators. The total number of generators is cubic in N_f :

$$2 + 3N_f^2 + 2N_f^3 . \quad (4.20)$$

Dimensions from plethystic logarithms: A trick. In the above, we computed dimensions of representations for relations using the following technique. For definiteness, let us consider the relations at order $s^2 t^3 \tilde{t}^3$. We know that the dimension $D(N_f)$ of the $SU(N_f) \times SU(N_f)$ representation $[0, 0, 1, 0, \dots, 0; 0, \dots, 0, 1, 0, 0] + [1, 1, 0, \dots, 0; 0, \dots, 0, 1, 0, 0] + [0, 0, 1, 0, \dots, 0; 0, \dots, 0, 1, 1] + [1, 1, 0, \dots, 0; 0, \dots, 0, 1, 1]$ must be a polynomial of order 6 in N_f :

$$D(N_f) = \sum_{k=0}^6 a_k N_f^k . \quad (4.21)$$

Observe that we can determine the unknowns a_0, \dots, a_6 from the 7 data points which come from the coefficients of $s^2 t^3 \tilde{t}^3$ in (4.14) for $N_f = 0, \dots, 6$ (where the Hilbert series for $N_f = 0$ can be obtained by setting $t = \tilde{t} = 0$). Solving the following 7 equations simultaneously

$$\begin{aligned} D(0) &= 0, & D(1) &= 0, & D(2) &= 4, & D(3) &= 81, \\ D(4) &= 576, & D(5) &= 2500, & D(6) &= 8100, \end{aligned} \quad (4.22)$$

we find that

$$\begin{aligned} a_0 &= 0, & a_1 &= 0, & a_2 &= 0, & a_3 &= 0, \\ a_4 &= \frac{1}{4}, & a_5 &= -\frac{1}{2}, & a_6 &= \frac{1}{4}. \end{aligned} \quad (4.23)$$

Substituting back to (4.21), we arrive at

$$D(N_f) = \frac{1}{4} N_f^4 (N_f - 1)^2 . \quad (4.24)$$

4.1.3 The $SU(4)$ Gauge Group

Let us examine the $SU(4)$ theory with N_f flavours and 1 adjoint matter. We have N_f chiral superfields transforming in the fundamental representation, N_f chiral superfields transforming in the antifundamental representation, and 1 chiral superfield transforming in the adjoint representation. Therefore, we can apply the Molien–Weyl formula to

our theory as follows:

$$\begin{aligned}
g^{(N_f, SU(4))} &= \frac{1}{(2\pi i)^3} \oint_{|z_1|=1} \frac{dz_1}{z_1} \oint_{|z_2|=1} \frac{dz_2}{z_2} \oint_{|z_3|=1} \frac{dz_3}{z_3} \left(1 - \frac{z_1^2}{z_2}\right) \left(1 - \frac{z_1 z_2}{z_3}\right) (1 - z_1 z_3) \left(1 - \frac{z_2^2}{z_1 z_3}\right) \times \\
&\quad \left(1 - \frac{z_2 z_3}{z_1}\right) \left(1 - \frac{z_3^2}{z_2}\right) \text{PE} [N_f[1, 0, 0]t + N_f[0, 0, 1]\tilde{t} + [1, 0, 1]s] \\
&= \frac{1}{(2\pi i)^3} \oint_{|z_1|=1} \frac{dz_1}{z_1} \oint_{|z_2|=1} \frac{dz_2}{z_2} \oint_{|z_3|=1} \frac{dz_3}{z_3} \left(1 - \frac{z_1^2}{z_2}\right) \left(1 - \frac{z_1 z_2}{z_3}\right) (1 - z_1 z_3) \left(1 - \frac{z_2^2}{z_1 z_3}\right) \times \\
&\quad \left(1 - \frac{z_2 z_3}{z_1}\right) \left(1 - \frac{z_3^2}{z_2}\right) \times \frac{1}{\left((1 - tz_1) \left(1 - t\frac{z_2}{z_1}\right) \left(1 - t\frac{z_3}{z_2}\right) \left(1 - \frac{t}{z_3}\right)\right)^{N_f}} \times \\
&\quad \frac{1}{\left(\left(1 - \frac{\tilde{t}}{z_1}\right) \left(1 - \tilde{t}\frac{z_1}{z_2}\right) \left(1 - \tilde{t}\frac{z_2}{z_3}\right) (1 - \tilde{t}z_3)\right)^{N_f}} \times \\
&\quad \frac{1}{(1 - s)^3 \left(1 - \frac{sz_1^2}{z_2}\right) \left(1 - \frac{sz_2}{z_1}\right) \left(1 - \frac{sz_2}{z_3}\right) \left(1 - \frac{s}{z_1 z_3}\right) \left(1 - \frac{sz_1}{z_2 z_3}\right) \left(1 - \frac{sz_1 z_2}{z_3}\right) \left(1 - \frac{sz_2^2}{z_1 z_3}\right)} \times \\
&\quad \frac{1}{(1 - sz_1 z_3) \left(1 - \frac{sz_1 z_3}{z_2}\right) \left(1 - \frac{sz_3}{z_1 z_2}\right) \left(1 - \frac{sz_2 z_3}{z_1}\right) \left(1 - \frac{sz_3^2}{z_2}\right)} . \tag{4.25}
\end{aligned}$$

Applying the residue theorem, we find that

$$\begin{aligned}
g^{(1, SU(4))}(s, t, \tilde{t}) &= \frac{1 - s^{12}t^4\tilde{t}^4}{(1 - s^2)(1 - s^3)(1 - s^4)(1 - t\tilde{t})(1 - st\tilde{t})(1 - s^2t\tilde{t})(1 - s^3t\tilde{t})(1 - s^6t^4)(1 - s^6\tilde{t}^4)} \\
&= 1 + s^2 + s^3 + 2s^4 + s^5 + 3s^6 + 2s^7 + 4s^8 + 3s^9 + 5s^{10} + 4s^{11} + 7s^{12} + s^6t^4 + \\
&\quad s^8t^4 + s^9t^4 + 2s^{10}t^4 + s^{11}t^4 + 3s^{12}t^4 + t\tilde{t} + st\tilde{t} + 2s^2t\tilde{t} + 3s^3t\tilde{t} + 4s^4t\tilde{t} + \\
&\quad 5s^5t\tilde{t} + 7s^6t\tilde{t} + 8s^7t\tilde{t} + 10s^8t\tilde{t} + 12s^9t\tilde{t} + 14s^{10}t\tilde{t} + 16s^{11}t\tilde{t} + 19s^{12}t\tilde{t} + \\
&\quad t^2\tilde{t}^2 + st^2\tilde{t}^2 + 3s^2t^2\tilde{t}^2 + 4s^3t^2\tilde{t}^2 + 7s^4t^2\tilde{t}^2 + 8s^5t^2\tilde{t}^2 + 13s^6t^2\tilde{t}^2 + 14s^7t^2\tilde{t}^2 + \\
&\quad 20s^8t^2\tilde{t}^2 + 22s^9t^2\tilde{t}^2 + 29s^{10}t^2\tilde{t}^2 + 31s^{11}t^2\tilde{t}^2 + 40s^{12}t^2\tilde{t}^2 + t^3\tilde{t}^3 + st^3\tilde{t}^3 + 3s^2t^3\tilde{t}^3 + \\
&\quad 5s^3t^3\tilde{t}^3 + 8s^4t^3\tilde{t}^3 + 11s^5t^3\tilde{t}^3 + 17s^6t^3\tilde{t}^3 + 21s^7t^3\tilde{t}^3 + 28s^8t^3\tilde{t}^3 + 35s^9t^3\tilde{t}^3 + \\
&\quad 43s^{10}t^3\tilde{t}^3 + 51s^{11}t^3\tilde{t}^3 + 62s^{12}t^3\tilde{t}^3 + s^6\tilde{t}^4 + s^8\tilde{t}^4 + s^9\tilde{t}^4 + 2s^{10}\tilde{t}^4 + s^{11}\tilde{t}^4 + 3s^{12}\tilde{t}^4 + \\
&\quad t^4\tilde{t}^4 + st^4\tilde{t}^4 + 3s^2t^4\tilde{t}^4 + 5s^3t^4\tilde{t}^4 + 9s^4t^4\tilde{t}^4 + 12s^5t^4\tilde{t}^4 + 20s^6t^4\tilde{t}^4 + 25s^7t^4\tilde{t}^4 + 36s^8t^4\tilde{t}^4 + \\
&\quad 44s^9t^4\tilde{t}^4 + 58s^{10}t^4\tilde{t}^4 + 68s^{11}t^4\tilde{t}^4 + 87s^{12}t^4\tilde{t}^4 + O(s^{13})O(t^5)O(\tilde{t}^5), \\
g^{(2, SU(4))}(s, t, \tilde{t}) &= 1 + s^2 + s^3 + 2s^4 + s^5 + 3s^6 + 2s^7 + 4s^8 + s^2t^4 + 3s^3t^4 + 5s^4t^4 + 7s^5t^4 + 14s^6t^4 + \\
&\quad 14s^7t^4 + 22s^8t^4 + 4t\tilde{t} + 4st\tilde{t} + 8s^2t\tilde{t} + 12s^3t\tilde{t} + 16s^4t\tilde{t} + 20s^5t\tilde{t} + 28s^6t\tilde{t} + 32s^7t\tilde{t} + \\
&\quad 40s^8t\tilde{t} + 10t^2\tilde{t}^2 + 16st^2\tilde{t}^2 + 36s^2t^2\tilde{t}^2 + 57s^3t^2\tilde{t}^2 + 87s^4t^2\tilde{t}^2 + 114s^5t^2\tilde{t}^2 + \\
&\quad 163s^6t^2\tilde{t}^2 + 196s^7t^2\tilde{t}^2 + 255s^8t^2\tilde{t}^2 + 20t^3\tilde{t}^3 + 40st^3\tilde{t}^3 + 100s^2t^3\tilde{t}^3 + 180s^3t^3\tilde{t}^3 + \\
&\quad 296s^4t^3\tilde{t}^3 + 432s^5t^3\tilde{t}^3 + 624s^6t^3\tilde{t}^3 + 816s^7t^3\tilde{t}^3 + 1064s^8t^3\tilde{t}^3 + s^2\tilde{t}^4 + 3s^3\tilde{t}^4 + \\
&\quad 5s^4\tilde{t}^4 + 7s^5\tilde{t}^4 + 14s^6\tilde{t}^4 + 14s^7\tilde{t}^4 + 22s^8\tilde{t}^4 + 35t^4\tilde{t}^4 + 80st^4\tilde{t}^4 + 215s^2t^4\tilde{t}^4 + \\
&\quad 425s^3t^4\tilde{t}^4 + 759s^4t^4\tilde{t}^4 + 1193s^5t^4\tilde{t}^4 + 1816s^6t^4\tilde{t}^4 + 2508s^7t^4\tilde{t}^4 + 3404s^8t^4\tilde{t}^4 + \\
&\quad O(s^9)O(t^5)O(\tilde{t}^5), \\
g^{(3, SU(4))}(s, t, \tilde{t}) &= 1 + s^2 + s^3 + 2s^4 + s^5 + 3s^6 + 2s^7 + 4s^8 + 3st^4 + 9s^2t^4 + 21s^3t^4 + 33s^4t^4 + 48s^5t^4 + \\
&\quad 75s^6t^4 + 90s^7t^4 + 123s^8t^4 + 9t\tilde{t} + 9st\tilde{t} + 18s^2t\tilde{t} + 27s^3t\tilde{t} + 36s^4t\tilde{t} + 45s^5t\tilde{t} + 63s^6t\tilde{t} + \\
&\quad 72s^7t\tilde{t} + 90s^8t\tilde{t} + 45t^2\tilde{t}^2 + 81st^2\tilde{t}^2 + 171s^2t^2\tilde{t}^2 + 279s^3t^2\tilde{t}^2 + 414s^4t^2\tilde{t}^2 + \\
&\quad 558s^5t^2\tilde{t}^2 + 774s^6t^2\tilde{t}^2 + 954s^7t^2\tilde{t}^2 + 1215s^8t^2\tilde{t}^2 + 165t^3\tilde{t}^3 + 405st^3\tilde{t}^3 + 974s^2t^3\tilde{t}^3 + \\
&\quad 1787s^3t^3\tilde{t}^3 + 2920s^4t^3\tilde{t}^3 + 4297s^5t^3\tilde{t}^3 + 6103s^6t^3\tilde{t}^3 + 8044s^7t^3\tilde{t}^3 + 10414s^8t^3\tilde{t}^3 + \\
&\quad 3s\tilde{t}^4 + 9s^2\tilde{t}^4 + 21s^3\tilde{t}^4 + 33s^4\tilde{t}^4 + 48s^5\tilde{t}^4 + 75s^6\tilde{t}^4 + 90s^7\tilde{t}^4 + 123s^8\tilde{t}^4 + 495t^4\tilde{t}^4 + \\
&\quad 1485st^4\tilde{t}^4 + 3996s^2t^4\tilde{t}^4 + 8172s^3t^4\tilde{t}^4 + 14625s^4t^4\tilde{t}^4 + 23292s^5t^4\tilde{t}^4 + 34848s^6t^4\tilde{t}^4 + \\
&\quad 48465s^7t^4\tilde{t}^4 + 65043s^8t^4\tilde{t}^4 + O(s^9)O(t^5)O(\tilde{t}^5), \\
g^{(4, SU(4))}(s, t, \tilde{t}) &= 1 + s^2 + s^3 + 2s^4 + s^5 + 3s^6 + 2s^7 + 4s^8 + t^4 + 15st^4 + 36s^2t^4 + 76s^3t^4 + 117s^4t^4 + \\
&\quad 171s^5t^4 + 248s^6t^4 + 312s^7t^4 + 409s^8t^4 + 16t\tilde{t} + 16st\tilde{t} + 32s^2t\tilde{t} + 48s^3t\tilde{t} + 64s^4t\tilde{t} + \\
&\quad 80s^5t\tilde{t} + 112s^6t\tilde{t} + 128s^7t\tilde{t} + 160s^8t\tilde{t} + 136t^2\tilde{t}^2 + 256st^2\tilde{t}^2 + 528s^2t^2\tilde{t}^2 + 868s^3t^2\tilde{t}^2 + \\
&\quad 1276s^4t^2\tilde{t}^2 + 1736s^5t^2\tilde{t}^2 + 2380s^6t^2\tilde{t}^2 + 2960s^7t^2\tilde{t}^2 + 3740s^8t^2\tilde{t}^2 + 816t^3\tilde{t}^3 + \\
&\quad 2176st^3\tilde{t}^3 + 5152s^2t^3\tilde{t}^3 + 9488s^3t^3\tilde{t}^3 + 15424s^4t^3\tilde{t}^3 + 22720s^5t^3\tilde{t}^3 + 32032s^6t^3\tilde{t}^3 + \\
&\quad 42288s^7t^3\tilde{t}^3 + 54560s^8t^3\tilde{t}^3 + \tilde{t}^4 + 15s\tilde{t}^4 + 36s^2\tilde{t}^4 + 76s^3\tilde{t}^4 + 117s^4\tilde{t}^4 + 171s^5\tilde{t}^4 + \\
&\quad 248s^6\tilde{t}^4 + 312s^7\tilde{t}^4 + 409s^8\tilde{t}^4 + 3876t^4\tilde{t}^4 + 13055st^4\tilde{t}^4 + 35171s^2t^4\tilde{t}^4 + 72451s^3t^4\tilde{t}^4 + \\
&\quad 129257s^4t^4\tilde{t}^4 + 205641s^5t^4\tilde{t}^4 + 305193s^6t^4\tilde{t}^4 + 423847s^7t^4\tilde{t}^4 + 565889s^8t^4\tilde{t}^4 + \\
&\quad O(s^9)O(t^5)O(\tilde{t}^5). \quad - 20 -
\end{aligned}$$

We emphasise that these results with explicit expansion up to order $s^8 t^4 \tilde{t}^4$, albeit rather lengthy, turn out to be essential for analyses of the generators and their basic relations in the chiral ring.

Plethystic logarithms. We shall calculate plethystic logarithms of generating functions using formula (2.16):

$$\begin{aligned}
\text{PL} \left[g^{(1, SU(4))}(s, t, \tilde{t}) \right] &= s^2 + s^3 + s^4 + t\tilde{t} + st\tilde{t} + s^2 t\tilde{t} + s^3 t\tilde{t} + s^6 t^4 + s^6 \tilde{t}^4 - s^{12} t^4 \tilde{t}^4, \\
\text{PL} \left[g^{(2, SU(4))}(s, t, \tilde{t}) \right] &= s^2 + s^3 + s^4 + s^2 t^4 + 3s^3 t^4 + 4s^4 t^4 + 3s^5 t^4 + 5s^6 t^4 + 4t\tilde{t} + 4st\tilde{t} \\
&\quad + 4s^2 t\tilde{t} + 4s^3 t\tilde{t} - s^3 t^2 \tilde{t}^2 - s^4 t^2 \tilde{t}^2 - s^5 t^2 \tilde{t}^2 - s^6 t^2 \tilde{t}^2 + s^2 \tilde{t}^4 + 3s^3 \tilde{t}^4 \\
&\quad + 4s^4 \tilde{t}^4 + 3s^5 \tilde{t}^4 + 5s^6 \tilde{t}^4 - s^4 t^4 \tilde{t}^4 - 6s^5 t^4 \tilde{t}^4 - 17s^6 t^4 \tilde{t}^4 - 30s^7 t^4 \tilde{t}^4 \\
&\quad - 44s^8 t^4 \tilde{t}^4 + O(s^9)O(t^5)O(\tilde{t}^5), \\
\text{PL} \left[g^{(3, SU(4))}(s, t, \tilde{t}) \right] &= s^2 + s^3 + s^4 + 3st^4 + 9s^2 t^4 + 18s^3 t^4 + 21s^4 t^4 + 15s^5 t^4 + 15s^6 t^4 \\
&\quad + 9t\tilde{t} + 9st\tilde{t} + 9s^2 t\tilde{t} + 9s^3 t\tilde{t} - 9s^3 t^2 \tilde{t}^2 - 9s^4 t^2 \tilde{t}^2 - 9s^5 t^2 \tilde{t}^2 - 9s^6 t^2 \tilde{t}^2 \\
&\quad - s^2 t^3 \tilde{t}^3 - s^3 t^3 \tilde{t}^3 - s^4 t^3 \tilde{t}^3 + 17s^6 t^3 \tilde{t}^3 + 17s^7 t^3 \tilde{t}^3 + 17s^8 t^3 \tilde{t}^3 + 3s\tilde{t}^4 \\
&\quad + 9s^2 \tilde{t}^4 + 18s^3 \tilde{t}^4 + 21s^4 \tilde{t}^4 + 15s^5 \tilde{t}^4 + 15s^6 \tilde{t}^4 - 9s^2 t^4 \tilde{t}^4 - 54s^3 t^4 \tilde{t}^4 \\
&\quad - 189s^4 t^4 \tilde{t}^4 - 441s^5 t^4 \tilde{t}^4 - 783s^6 t^4 \tilde{t}^4 - 1107s^7 t^4 \tilde{t}^4 - 1251s^8 t^4 \tilde{t}^4 \\
&\quad + O(s^9)O(t^5)O(\tilde{t}^5), \\
\text{PL} \left[g^{(4, SU(4))}(s, t, \tilde{t}) \right] &= s^2 + s^3 + s^4 + t^4 + 15st^4 + 35s^2 t^4 + 60s^3 t^4 + 65s^4 t^4 + 45s^5 t^4 + 35s^6 t^4 \\
&\quad + 16t\tilde{t} + 16st\tilde{t} + 16s^2 t\tilde{t} + 16s^3 t\tilde{t} - 36s^3 t^2 \tilde{t}^2 - 36s^4 t^2 \tilde{t}^2 - 36s^5 t^2 \tilde{t}^2 \\
&\quad - 36s^6 t^2 \tilde{t}^2 - 16s^2 t^3 \tilde{t}^3 - 16s^3 t^3 \tilde{t}^3 - 16s^4 t^3 \tilde{t}^3 + 176s^6 t^3 \tilde{t}^3 + 176s^7 t^3 \tilde{t}^3 \\
&\quad + 176s^8 t^3 \tilde{t}^3 + \tilde{t}^4 + 15s\tilde{t}^4 + 35s^2 \tilde{t}^4 + 60s^3 \tilde{t}^4 + 65s^4 \tilde{t}^4 + 45s^5 \tilde{t}^4 + 35s^6 \tilde{t}^4 \\
&\quad - t^4 \tilde{t}^4 - 31st^4 \tilde{t}^4 - 296s^2 t^4 \tilde{t}^4 - 1170s^3 t^4 \tilde{t}^4 - 3124s^4 t^4 \tilde{t}^4 - 5912s^5 t^4 \tilde{t}^4 \\
&\quad - 9243s^6 t^4 \tilde{t}^4 - 11704s^7 t^4 \tilde{t}^4 - 12106s^8 t^4 \tilde{t}^4 + O(s^9)O(t^5)O(\tilde{t}^5).
\end{aligned}$$

Generators of the GIOs. Armed with plethystic logarithms, we can write down the generators of the GIOs.

$$\begin{aligned}
\text{Casimir invariants : } s^k &\rightarrow u_k = \text{Tr}(\phi^k), \quad k = 2, 3, 4 \\
&\quad [0, \dots, 0; 0, \dots, 0] \quad 1 \text{ dimensional}, \\
\text{Mesons : } t\tilde{t} &\rightarrow M_j^i = Q_a^i \tilde{Q}_j^a \\
&\quad [1, 0, \dots, 0; 0, \dots, 0, 1] \quad N_f^2 \text{ dimensional}, \\
\text{Adjoint mesons : } s^l t\tilde{t} &\rightarrow (A_l)^i_j = Q_a^i (\phi^l)_b^a \tilde{Q}_j^b, \quad l = 1, 2, 3 \\
&\quad [1, 0, \dots, 0; 0, \dots, 0, 1] \quad N_f^2 \text{ dimensional}, \\
\text{Baryons : } t^4 &\rightarrow B^{i_1 i_2 i_3 i_4} = \epsilon^{a_1 a_2 a_3 a_4} Q_{a_1}^{i_1} Q_{a_2}^{i_2} Q_{a_3}^{i_3} Q_{a_4}^{i_4} \\
&\quad [0, 0, 0, 1, 0, \dots, 0; 0, \dots, 0] \quad \binom{N_f}{4} \text{ dimensional}, \\
\text{Antibaryons : } \tilde{t}^4 &\rightarrow \text{similar expression to baryons with } Q \rightarrow \tilde{Q} \\
&\quad [0, \dots, 0; 0, \dots, 0, 1, 0, 0, 0] \quad \binom{N_f}{4} \text{ dimensional}.
\end{aligned}$$

In addition, we have **adjoint baryons** (in a similar fashion to the case of $N_c = 3$):

$$\begin{aligned}
st^4 &\rightarrow \mathcal{B}_{0,0,0,1} = \epsilon QQQP_1 \\
&\quad [1, 0, 1, 0, \dots, 0; 0, \dots, 0]^\dagger \\
s^2t^4 &\rightarrow \mathcal{B}_{0,0,0,2} = \epsilon QQQP_2, \mathcal{B}_{0,0,1,1} = \epsilon QQP_1P_1 \\
&\quad [0, 2, 0, \dots, 0; 0, \dots, 0] + [1, 0, 1, 0, \dots, 0; 0, \dots, 0]^\ddagger \\
s^3t^4 &\rightarrow \mathcal{B}_{0,0,0,3} = \epsilon QQQP_3, \mathcal{B}_{0,0,1,2} = \epsilon QQP_1P_2, \mathcal{B}_{0,1,1,1} = \epsilon QP_1P_1P_1 \\
&\quad [2, 1, 0, \dots, 0; 0, \dots, 0] + [1, 0, 1, 0, \dots, 0; 0, \dots, 0] \\
s^4t^4 &\rightarrow \mathcal{B}_{0,0,0,4} = \epsilon QQQP_4, \mathcal{B}_{0,0,1,3} = \epsilon QQP_1P_3, \mathcal{B}_{0,0,2,2} = \epsilon QQP_2P_2 \\
&\quad \mathcal{B}_{0,1,1,2} = \epsilon QP_1P_1P_2, \mathcal{B}_{1,1,1,1} = \epsilon P_1P_1P_1P_1 \\
&\quad [2, 1, 0, \dots, 0; 0, \dots, 0] + [0, 2, 0, \dots, 0; 0, \dots, 0] \\
s^5t^4 &\rightarrow \mathcal{B}_{0,0,0,5} = \epsilon QQQP_5, \mathcal{B}_{0,0,1,4} = \epsilon QQP_1P_4, \mathcal{B}_{0,0,2,3} = \epsilon QQP_2P_2 \\
&\quad \mathcal{B}_{0,1,1,3} = \epsilon QP_1P_1P_3, \mathcal{B}_{0,1,2,2} = \epsilon QP_1P_2P_2, \mathcal{B}_{1,1,1,2} = \epsilon P_1P_1P_1P_2 \\
&\quad [2, 1, 0, \dots, 0; 0, \dots, 0] \\
s^6t^4 &\rightarrow \mathcal{B}_{0,0,0,6} = \epsilon QQQP_6, \mathcal{B}_{0,0,1,5} = \epsilon QQP_1P_5, \mathcal{B}_{0,0,3,3} = \epsilon QQP_3P_3 \\
&\quad \mathcal{B}_{0,1,1,4} = \epsilon QP_1P_1P_4, \mathcal{B}_{0,1,2,3} = \epsilon QP_1P_2P_3, \mathcal{B}_{0,2,2,2} = \epsilon QP_2P_2P_2 \\
&\quad \mathcal{B}_{1,1,1,3} = \epsilon P_1P_1P_1P_3, \mathcal{B}_{1,1,2,2} = \epsilon P_1P_1P_2P_2, \mathcal{B}_{0,0,2,4} = \epsilon QQP_2P_4 \\
&\quad [4, 0, \dots, 0; 0, \dots, 0]
\end{aligned}$$

where $(P_m)_a^i = \phi_a^{b_1} \phi_{b_1}^{b_2} \dots \phi_{b_{m-1}}^{b_m} Q_{b_m}^i$, and the subscript of \mathcal{B} indicates the partition of the power of s in the adjoint baryon. In the above, we suppressed the indices with the understanding that each epsilon tensor is contracted over all colour indices. Moreover, we have **adjoint antibaryons** which transform in the conjugate representations of adjoint baryons.

As for the case of $SU(3)$ gauge group, we emphasise that the representations written above are *not* the ones in which the generators transform; however, they are the ones in which the relations have already been taken into account. For example,

[†]**The generator at order st^4 .** Note that the generator $\mathcal{B}_{0,0,0,1}^{i_1 i_2 i_3 j_1} = \epsilon^{a_1 a_2 a_3 b_1} Q_{a_1}^{i_1} Q_{a_2}^{i_2} Q_{a_3}^{i_3} (P_1)_{b_1}^{j_1}$ satisfies a relation:

$$\mathcal{B}_{0,0,0,1}^{[i_1 i_2 i_3 j_1]} = 0, \quad (4.27)$$

where the square bracket denotes an antisymmetrisation without a normalisation factor. This means that the completely antisymmetric part, which transforms in the $SU(N_f)$ representation $[0,0,0,1,0,\dots,0]$, vanishes. Note that we can construct the generator by considering the following $SU(N_f)$ tensor product:

$$[0, 0, 1, 0, \dots, 0] \times [1, 0, \dots, 0] = [1, 0, 1, 0, \dots, 0] + [0, 0, 0, 1, 0, \dots, 0].$$

Therefore, after taking (4.27) into account, we conclude that $\mathcal{B}_{0,0,0,1}^{i_1 i_2 i_3 j_1}$ transforms in the $SU(N_f) \times SU(N_f)$ representation $[1,0,1,0, \dots, 0; 0, \dots, 0]$, as stated in the list above.

‡Two generators at order $s^2 t^4$. We can construct $\mathcal{B}_{0,0,1,1}, \mathcal{B}_{0,0,0,2}$ by considering the $SU(N_f)$ tensor products:

$$\begin{aligned} [0, 1, 0, \dots, 0]^2 &= [0, 2, 0, \dots, 0] + [1, 0, 1, 0, \dots, 0] + [0, 0, 0, 1, 0, \dots, 0] , \\ [0, 0, 1, 0, \dots, 0] \times [1, 0, \dots, 0] &= [1, 0, 1, 0, \dots, 0] + [0, 0, 0, 1, 0, \dots, 0] . \end{aligned}$$

They are however subject to the relations:

$$\mathcal{B}_{0,0,0,2}^{[ijkm]} = 6 \left(\mathcal{B}_{0,0,0,2}^{i[jkm]} + \mathcal{B}_{0,0,1,1}^{i[jkm]} \right) , \quad (4.28)$$

$$\mathcal{B}_{0,0,0,2}^{[ijkm]} = -3 \mathcal{B}_{0,0,1,1}^{[ijkm]} , \quad (4.29)$$

which transform respectively in the $SU(N_f)$ representations $[1, 0, 1, 0, \dots, 0] + [0, 0, 0, 1, 0, \dots, 0]$, $[0, 0, 0, 1, 0, \dots, 0]$. Therefore, we are left with the global $SU(N_f) \times SU(N_f)$ representation $[0, 2, 0, \dots, 0; 0, \dots, 0] + [1, 0, 1, 0, \dots, 0; 0, \dots, 0]$, as stated in the above list.

Total number of generators. Using the trick mentioned in the previous subsection, we find that the total number of generators is

$$3 + 4N_f^2 + 2N_f^4 . \quad (4.30)$$

From (4.7), (4.20) and (4.30), we establish the following observation⁶:

Observation 4.1. *The total number of generators in the $SU(N_c)$ theory with N_f fundamental chiral superfields and 1 adjoint chiral superfield is of order $N_f^{N_c}$.*

Note that this is substantially higher than the theory with no adjoints.

A comment on representations. From a number of examples in the cases of $SU(3)$ and $SU(4)$ gauge groups, we establish the following observations:

Observation 4.2. *Any adjoint baryon of the form $\mathcal{B}_{\alpha_1, \dots, \alpha_{N_c}}$ (with $0 \leq \alpha_1 \leq \dots \leq \alpha_{N_c} \leq N_c$) exists in the theory as a generator. Note that $N_A \equiv \alpha_1 + \dots + \alpha_{N_c}$ is the total number of adjoint fields appearing in this particular adjoint baryon. It satisfies the bounds: $0 \leq N_A \leq \frac{1}{2} N_c (N_c - 1)$.*

Observation 4.3. *Whenever there is more than one way in partitioning adjoint fields into an adjoint baryon, there exists a relation between those options. The relation must transform in such a way that it cancels some representations associated with the generators, so that the leftover agrees with plethystic logarithms.*

⁶From now on, we use the word **Observation** to refer to a strong conjecture which can be deduced, in a consistent manner, from a number of non-trivial results presented earlier.

4.2 Adjoint Baryons: A Combinatorial Problem of Partitions

From a combinatorial point of view, Observation 4.2 suggests that an adjoint baryon is simply a partition of the N_A objects into N_c slots (without distinction between the slots). This leads to an interesting problem: For given N_A and N_c , how many adjoint baryons can be constructed?

The partition function. This problem can be elegantly solved using a partition function (Hilbert series). Suppose that the number of slots N_c is held fixed. Let t be a fugacity conjugate to the number of adjoint fields N_A . The required partition function is

$$\mathcal{Z}_{N_c}(t) = \sum_{N_A=0}^{\infty} a_{N_A, N_c} t^{N_A} , \quad (4.31)$$

where a_{N_A, N_c} is the number of adjoint baryons which can be constructed for given N_A and N_c . We can write the partition \mathcal{Z} in another way as follows. Let n_i be the number of slots which contain i adjoint fields. It is then easy to see that $n_1 + 2n_2 + \dots + N_c n_{N_c}$ is the total number of adjoint fields. We can therefore write

$$\mathcal{Z}_{N_c}(t) = \sum_{\{n_i\}} t^{n_1 + 2n_2 + \dots + N_c n_{N_c}} = \frac{1}{(1-t)(1-t^2)\dots(1-t^{N_c})} = \prod_{j=1}^{N_c} \frac{1}{1-t^j} . \quad (4.32)$$

This formula is also known as a partition function of N_c bosonic one-dimensional harmonic oscillators [3]. Equating (4.31) and (4.32), we find that

$$\sum_{N_A=0}^{\infty} a_{N_A, N_c} t^{N_A} = \prod_{j=1}^{N_c} \frac{1}{1-t^j} . \quad (4.33)$$

Thus, the number of adjoint baryons a_{N_A, N_c} is given by the coefficient of t^{N_A} in the power series of the product $\prod_{j=1}^{N_c} \frac{1}{1-t^j}$. In other words,

$$a_{N_A, N_c} = \frac{1}{2\pi i} \oint_{|t|=1} \frac{dt}{t^{N_A+1}} \prod_{j=1}^{N_c} \frac{1}{1-t^j} . \quad (4.34)$$

We note that this result is correct for any $N_A \geq 0$ but we are particularly interested in the case of $0 \leq N_A \leq \binom{N_c}{2}$.

Example: $N_c = 3$. The power series of the last expression in (4.33) is given by

$$\prod_{j=1}^3 \frac{1}{1-t^j} = 1 + t + 2t^2 + 3t^3 + \dots . \quad (4.35)$$

Therefore, for 0, 1, 2 and 3 adjoint fields, we can construct 1, 1, 2 and 3 adjoint baryons, respectively. This agrees with the earlier results above Equation (4.15) for the $SU(3)$ gauge group.

Example: $N_c = 4$. The power series of the last expression in (4.33) is given by

$$\prod_{j=1}^4 \frac{1}{1-t^j} = 1 + t + 2t^2 + 3t^3 + 5t^4 + 6t^5 + 9t^6 + \dots \quad (4.36)$$

Therefore, for 0, 1, 2, 3, 4, 5 and 6 adjoint fields, we can construct 1, 1, 2, 3, 5, 6 and 9 adjoint baryons, respectively. This agrees with the earlier results above Equation (4.27) for the $SU(4)$ gauge group.

4.2.1 An Asymptotic Formula

In this subsection, we derive an asymptotic formula for (4.34). Since the upper bound of the number of adjoint fields N_A is of order N_c^2 , we consider the limit of large N_c and fixed ratio

$$x \equiv \frac{N_A}{N_c^2} \ , \quad (4.37)$$

where x is of order one⁷. Recall from (4.34) that

$$a_{N_A, N_c} = \frac{1}{2\pi i} \oint_{|t|=1} \frac{dt}{t^{N_A+1}} \mathcal{Z}_{N_c}(t) \ . \quad (4.38)$$

The main contribution to the integral comes from $t \lesssim 1$, and so we set

$$t = 1 - \frac{a}{N_c} \ , \quad (4.39)$$

where a is a number of order 1. The behavior of t with N_c is determined by observing that this is a 2 dimensional partition problem in which the scaling of $1 - t$ goes like $N_A^{-1/2}$.

⁷For adjoint baryons, we are interested in the range $0 \leq x \lesssim \frac{1}{2}$.

Asymptotic formula for $\mathcal{Z}_{N_c}(t)$. We approximate $\mathcal{Z}_{N_c}(t)$ using the Euler–Maclaurin formula⁸ as follows:

$$\begin{aligned}
\log \mathcal{Z}_{N_c}(t) &= \log \left(\prod_{j=1}^{N_c} \frac{1}{1-t^j} \right) = - \sum_{j=1}^{N_c} \log(1-t^j) \\
&\sim - \int_1^{N_c} \log(1-t^s) ds - \frac{1}{2} \log[(1-t)(1-t^{N_c})] \\
&= - \frac{\text{Li}_2(t) - \text{Li}_2(t^{N_c})}{\log t} - \frac{1}{2} \log[(1-t)(1-t^{N_c})] \\
&\sim N_c \left(\frac{\pi^2}{6a} - \frac{\text{Li}_2(e^{-a})}{a} \right) - \frac{1}{2} \log N_c + \left[\frac{1}{2} \log \left(\frac{a}{(e^a-1)(1-e^{-a})^a} \right) - \frac{\pi^2}{12} + \frac{a}{2} - 1 \right].
\end{aligned}$$

Thus, we find that

$$\mathcal{Z}_{N_c}(t) \sim \frac{C(a)}{\sqrt{N_c}} \exp \left[N_c \left(\frac{\pi^2}{6a} - \frac{\text{Li}_2(e^{-a})}{a} \right) \right], \quad (4.40)$$

where $C(a)$ is given by

$$C(a) = \sqrt{\frac{a}{(e^a-1)(1-e^{-a})^a}} \exp \left(-\frac{\pi^2}{12} + \frac{a}{2} - 1 \right). \quad (4.41)$$

The saddle point method. Substituting (4.40) into (4.34) and writing

$$\frac{1}{t^{N_A+1}} \sim \exp(-N_A \log t) \sim \exp[N_A(1-t)] = \exp \left(\frac{N_A}{N_c} a \right) = \exp(ax N_c),$$

we find that

$$a_{N_A, N_c} \sim \frac{1}{2\pi i} \oint_{\mathcal{C}} f(a, N_c) e^{N_c \phi(a)} da, \quad (4.42)$$

where the contour \mathcal{C} is taken to be a circle (with a small radius) enclosing the origin $a = 0$ in the anticlockwise direction and we define

$$f(a, N_c) = \frac{C(a)}{N_c^{3/2}}, \quad \phi(a) = ax + \frac{1}{a} \left(\frac{\pi^2}{6} - \text{Li}_2(e^{-a}) \right). \quad (4.43)$$

We deform the contour \mathcal{C} to a new contour passing through the location a_0 of the saddle point in the direction of steepest descent. We calculate a_0 from the relation $\phi'(a_0) = 0$:

$$x = \frac{1}{a_0^2} \left(\frac{\pi^2}{6} - \text{Li}_2(e^{-a_0}) \right) + \frac{1}{a_0} \log(1-e^{-a_0}) \equiv \xi(a_0), \quad (4.44)$$

x	a_0
1/2	1.405
1/4	2.273
1/8	3.468
1/10	3.934

Table 1: Numerical values of a_0 for various values of x .

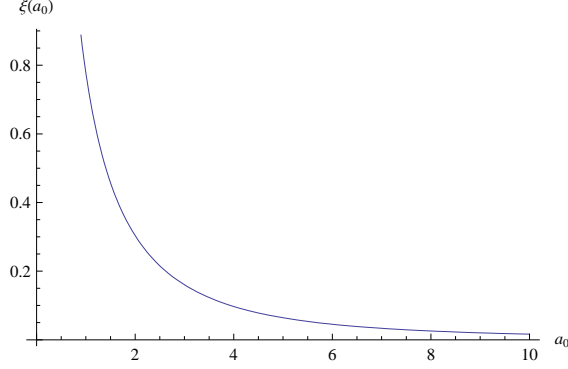


Figure 1: The graph of $\xi(a_0)$ against a_0 .

Since $\xi(a_0)$ is transcendental, it is difficult to obtain an analytical expression of a_0 in terms of x . However, given a numerical value of x , it is possible to determine the numerical value of a_0 (Table 1). The graph of $\xi(a_0)$ against a_0 is given in Figure 1.

We note that the second derivative $\phi''(a_0) > 0$, and so the steepest descent direction is parallel to the imaginary axis. We find that

$$\begin{aligned}
a_{N_A, N_c} &\sim \frac{1}{2\pi i} \int_{a_0 - i\epsilon}^{a_0 + i\epsilon} da \, f(a_0, N_c) \exp \left(N_c \phi(a_0) + \frac{1}{2} N_c \phi''(a_0) (a - a_0)^2 \right) \\
&= \frac{1}{2\pi} e^{N_c \phi(a_0)} f(a_0, N_c) \int_{-\epsilon}^{\epsilon} d\alpha \, e^{-\frac{1}{2} N_c \phi''(a_0) \alpha^2} \quad (a = a_0 + i\alpha) \\
&\sim \frac{1}{\pi} \frac{e^{N_c \phi(a_0)} f(a_0, N_c)}{\sqrt{2 N_c \phi''(a_0)}} \int_{-\infty}^{\infty} ds \, e^{-s^2} \quad \left(s = \alpha \sqrt{\frac{1}{2} N_c \phi''(a_0)} \right) \\
&\sim \frac{e^{N_c \phi(a_0)} f(a_0, N_c)}{\sqrt{2\pi N_c \phi''(a_0)}} . \tag{4.45}
\end{aligned}$$

⁸This formula states that $\sum_{n=a}^b f(n) \sim \int_a^b f(s) ds + \frac{1}{2} (f(a) + f(b))$.

Therefore, substituting (4.43) and (4.44) into (4.45), we have

$$a_{N_A, N_c} = \frac{F(a_0)}{N_c^2} \exp \left(N_c \left[\frac{2}{a_0} \left(\frac{\pi^2}{6} - \text{Li}_2(e^{-a_0}) \right) + \log(1 - e^{-a_0}) \right] \right) , \quad (4.46)$$

where

$$\begin{aligned} F(a_0) &= \frac{C(a_0)}{\sqrt{2\pi\phi''(a_0)}} , \\ \phi''(a_0) &= \frac{2}{a_0^3} \left(\frac{\pi^2}{6} - \text{Li}_2(e^{-a_0}) \right) + \frac{2}{a_0^2} \log(1 - e^{-a_0}) - \frac{1}{a_0(e^{a_0} - 1)} . \end{aligned} \quad (4.47)$$

Numerical values. We compare numerical values of the logarithm of (4.46) with the logarithm of the corresponding exact value in Table 2.

N_c	x	Logarithm of (4.46)	Logarithm of the exact a_{N_A, N_c}
50	1/2	74.640	75.333
100	1/2	157.58	158.28
50	1/4	53.180	53.945
100	1/4	114.06	114.84
100	1/8	80.008	80.837
1000	1/10	70.975	71.828

Table 2: Comparison of numerical values of the logarithm of (4.46) with the logarithm of the corresponding exact value.

4.3 The Canonical Free Energy

An immediate consequence of (3.1) is a general form of the unrefined generating function:

$$g^{(N_f, SU(N_c))}(t) = \frac{P(t)}{\prod_i (1 - t^{n_i})^{d_i}} , \quad (4.48)$$

where $P(t)$ is a palindromic polynomial with $P(1) \neq 0$, and the order of the pole $t = 1$ of $g^{(N_f, SU(N_c))}(t)$ is

$$\sum_i d_i = \dim \mathcal{M}_{(N_f, SU(N_c))} = 2N_f N_c . \quad (4.49)$$

We can define the **canonical free energy** of the system as

$$F(t) = -\log g^{(N_f, SU(N_c))}(t) . \quad (4.50)$$

It is easy to see from (4.48), (4.49) and (4.50) that in the large and N_c limit

$$F(t) \sim f(t) N_f N_c , \quad (4.51)$$

where $f(t)$ is some function of order 1. In other words, in this limit, the canonical free energy scales linearly with the dimension of the moduli space, which in turn is linear in both the number of colours and the number of flavours.

4.4 Complete Intersection Moduli Space

Having seen from a number of examples in preceding subsections that the moduli space of the theories with 1 flavour and 1 adjoint chiral multiplet is a complete intersection, we shall demonstrate that this statement is true for any $SU(N_c)$ gauge group.

Generators and relations. The generators of the theories with 1 flavour and 1 adjoint matter are

Casimir invariants :	s^k	$\rightarrow u_k = \text{Tr}(\phi^k), \quad k = 2, \dots, N_c$ 1 operator for each k ,
Meson :	$t\tilde{t}$	$\rightarrow M = Q_a \tilde{Q}^a$ 1 operator ,
Adjoint mesons :	$s^l t\tilde{t}$	$\rightarrow (A_l) = Q_a (\phi^l)_b^a \tilde{Q}^b, \quad l = 1, \dots, N_c - 1$ 1 operator for each l ,
Adjoint baryon :	$s^{(N_c-1)N_c/2} t^{N_c}$	$\rightarrow \mathcal{B}_{0,1,2,\dots,N_c-1} = \epsilon^{abc\dots d} Q_a (P_1)_b (P_2)_c \dots (P_{N_c-1})_d$ 1 operator ,
Adjoint antibaryon :	$s^{(N_c-1)N_c/2} \tilde{t}^{N_c}$	\rightarrow adjoint baryon with $Q \rightarrow \tilde{Q}$ 1 operator .

We see that there are altogether $2N_c + 1$ generators. Note that in all examples we checked the number of relations is 1. We therefore assume that there is precisely one basic relation at order $s^{(N_c-1)N_c} t^{N_c} \tilde{t}^{N_c}$.

Since the dimension of the moduli space (which is $2N_c$ from (3.1)) is equal to the number of generators (which is $2N_c + 1$) minus the number of basic relations (which is assumed to be 1), it gives a strong indication that the moduli space is a **complete intersection**.

General formula. As a consequence, we can write down a fully refined generating function for an arbitrary N_c as

$$g^{(1, SU(N_c))}(s, t, \tilde{t}) = \frac{1 - s^{(N_c-1)N_c} t^{N_c} \tilde{t}^{N_c}}{(1 - s^{(N_c-1)N_c/2} t^{N_c})(1 - s^{(N_c-1)N_c/2} \tilde{t}^{N_c}) \prod_{k=2}^{N_c} (1 - s^k) \prod_{l=0}^{N_c-1} (1 - s^l t\tilde{t})} . \quad (4.52)$$

4.4.1 A General Expression for The Relation

In §4.1.1, the relation for the case of $N_c = 2, N_f = 1$ is written explicitly in (4.9). It is interesting to find a general expression of the relation for any N_c (with $N_f = 1$).⁹ In this and only this subsection, we include the factor of $1/k$ into the Casimir invariant u_k , namely $u_k = \frac{1}{k} \text{Tr}(\phi^k)$.

The case of $N_c = 3$. Let us introduce the operators A_3 and A_4 in the usual way, *i.e.* $A_l = Q_a \phi^l \tilde{Q}^a$. Note that they can be written in terms of basic generators as

$$A_3 = u_3 A_0 + u_2 A_1, \quad A_4 = u_3 A_1 + u_2 A_2, \quad (4.53)$$

where A_0 denotes the meson M . Then, the relation can be written as

$$\mathcal{B}\tilde{\mathcal{B}} = A_0 A_2 A_4 - A_0 A_3^2 + 2A_1 A_2 A_3 - A_1^2 A_4 - A_2^3, \quad (4.54)$$

where \mathcal{B} and $\tilde{\mathcal{B}}$ respectively denote the baryon and antibaryon. The moduli space is $\mathbb{C}^9/\mathcal{I}_3$, where the ideal \mathcal{I}_3 is given by the 3 relations: (4.53) and (4.54).

The case of $N_c = 4$. We introduce the operators A_4, \dots, A_6 , which can be written in terms of basic generators as

$$\begin{aligned} A_4 &= u_4 A_0 - \frac{1}{2} u_2^2 A_0 + u_3 A_1 + u_2 A_2, \\ A_5 &= u_4 A_1 - \frac{1}{2} u_2^2 A_1 + u_3 A_2 + u_2 A_3, \\ A_6 &= u_2 u_4 A_0 - \frac{1}{2} u_2^3 A_0 + u_2 u_3 A_1 + \frac{1}{2} u_2^2 A_2 + u_4 A_2 + u_3 A_3. \end{aligned} \quad (4.55)$$

Then, the relation can be written as

$$\begin{aligned} \mathcal{B}\tilde{\mathcal{B}} &= A_0 A_2 A_4 A_6 - A_0 A_2 A_5^2 - A_0 A_3^2 A_6 + 2A_0 A_3 A_4 A_5 - A_0 A_4^3 - A_1^2 A_4 A_6 + A_1^2 A_5^2 \\ &\quad + 2A_1 A_2 A_3 A_6 - 2A_1 A_2 A_4 A_5 - 2A_1 A_3^2 A_5 + 2A_1 A_3 A_4^2 - A_2^3 A_6 + 2A_2^2 A_3 A_5 \\ &\quad + A_2^2 A_4^2 - 3A_2 A_3^2 A_4 + A_3^4. \end{aligned} \quad (4.56)$$

The moduli space is $\mathbb{C}^{12}/\mathcal{I}_4$, where the ideal \mathcal{I}_4 is given by the 4 relations: (4.55) and (4.56).

A general expression. We can generalise (4.54) and (4.56) to any number of colours. The relation can be written compactly as¹⁰

$$\mathcal{B}\tilde{\mathcal{B}} = \det \mathcal{A}, \quad (4.57)$$

⁹Special thanks to Nathan Seiberg, Kenneth Intriligator and Michael Douglas for discussions.

¹⁰We thank Michael Douglas for pointing out this elegant expression.

where $\mathcal{A}_{ij} = A_{i+j}$ and $0 \leq i, j \leq N_c - 1$.

It is interesting to examine this formula in the spacial case of $N_c = 2$. The adjoint baryon is given by $\mathcal{B} = \epsilon^{ab} Q_a^1 (P_1)_b^1 = A^{11}$, whereas the adjoint antibaryon is given by $\tilde{\mathcal{B}} = \epsilon_{ab} Q^{a2} (P_1)^{b2} = \epsilon_{ab} \epsilon^{ad} Q_d^2 \epsilon^{bc} (P_1)_c^2 = -\delta_b^d Q_d^2 \epsilon^{bc} (P_1)_c^2 = -\epsilon^{bc} Q_b^2 (P_1)_c^2 = -A^{22}$ (note the minus sign). Similarly, it is easy to see that $\det \mathcal{A} = A_0 A_2 - A_1^2 = M^{12} (\frac{1}{2} u M^{12}) - (A^{12})^2$. We thus correctly recover the formula (4.9).

The relations via Newton's formula and the Cayley-Hamilton theorem.

Consider the characteristic polynomial $p(x) = \det(x\mathbb{I} - \phi)$ of ϕ , which can be written as

$$p(x) = \sum_{j=0}^{N_c} x^j s_{N_c-j} , \quad (4.58)$$

with $s_0 = 1$ and $s_1 = -\text{Tr}(\phi) = 0$. Note that s_k is a symmetric polynomial $s_k = (-)^k \sum_{i_1 < \dots < i_k} \lambda_{i_1} \dots \lambda_{i_k}$ and $u_k = \frac{1}{k} \sum_i \lambda_i^k$, where λ 's are the eigenvalues of ϕ . It follows that the u 's and s 's are related by Newton's formula (see *e.g.*, [32, 33]):

$$n s_n + \sum_{j=1}^n j u_j s_{n-j} = 0 \quad (2 \leq n \leq N_c) . \quad (4.59)$$

Note that the case of $n = 1$ is trivial. Using the Cayley-Hamilton theorem, one obtains the matrix relation¹¹:

$$p(\phi) = \sum_{j=0}^{N_c} \phi^j s_{N_c-j} = 0 . \quad (4.60)$$

Multiplying (4.60) by ϕ^m (with $0 \leq m \leq N_c - 2$) and then by Q to the left and \tilde{Q} to the right, one obtains

$$\sum_{j=0}^{N_c} A_{j+m} s_{N_c-j} = 0 \quad (0 \leq m \leq N_c - 2) . \quad (4.61)$$

Note that (4.59) gives $N_c - 1$ relations between u 's and s 's and (4.61) gives $N_c - 1$ relations between A 's and s 's.

Counting generators and relations. There are altogether $4N_c - 1$ variables: A_l (with $l = 0, \dots, 2N_c - 2$); s_m, u_m (with $m = 2, \dots, N_c$); $\mathcal{B}, \tilde{\mathcal{B}}$. However, there are $2N_c - 1$ relations: $N_c - 1$ relations (4.59) between u 's and s 's; $N_c - 1$ relations (4.61) between A 's and s 's; and 1 relation (4.57). Thus, by the assumption of the moduli space being a complete intersection, we find that the dimension of the moduli space is $2N_c$, in agreement with the earlier result.

¹¹It should be noted that if we take a trace of (4.60), we obtain a special case $n = N_c$ of (4.59).

5. The $Sp(N_c)$ Gauge Groups

Let us turn to the $Sp(N_c)$ gauge theory¹² with $2N_f$ chiral superfields transforming in the fundamental representation (N_f flavours)¹³ and 1 chiral superfield transforming in the adjoint representation. The anomaly-free global symmetry of this theory [24] is $SU(2N_f) \times U(1) \times U(1)_R$.

5.1 Examples of Hilbert Series

Below we shall derive Hilbert series for various cases.

5.1.1 The $Sp(2)$ Gauge Group

Let us now examine the $Sp(2)$ gauge theory with $2N_f$ chiral multiplets in the fundamental representation and 1 chiral superfield in the adjoint representation. The Molien–Weyl formula for this theory is

$$\begin{aligned}
g^{(N_f, Sp(2))} &= \int_{Sp(2)} d\mu_{Sp(2)}(z_1, z_2) PE [2N_f[1, 0]t + [2, 0]s] \\
&= \oint_{|z_1|=1} \frac{dz_1}{2\pi i z_1} \oint_{|z_2|=1} \frac{dz_2}{2\pi i z_2} \frac{(1 - z_1^2)(1 - z_2)(1 - \frac{z_1^2}{z_2})(1 - \frac{z_2^2}{z_1^2})}{\left((1 - tz_1)(1 - t\frac{z_2}{z_1})(1 - t\frac{z_1}{z_2})(1 - t\frac{1}{z_1}) \right)^{2N_f}} \times \\
&\quad \frac{1}{(1 - s)^2(1 - sz_1^2)(1 - sz_2)(1 - s\frac{z_1^2}{z_2})(1 - s\frac{z_2^2}{z_1^2})(1 - s\frac{z_1^2}{z_2^2})(1 - s\frac{z_2}{z_1^2})(1 - s\frac{1}{z_2})(1 - s\frac{1}{z_1})}
\end{aligned} \tag{5.1}$$

¹²We shall use the notation where the rank of $Sp(n)$ is n and $Sp(1)$ is isomorphic to $SU(2)$.

¹³Note that the number of fundamental chiral multiplets must be even due to the global \mathbb{Z}_2 anomaly.

Applying the residue theorem, we can compute Hilbert series for various N_f :

$$\begin{aligned}
g^{(1,Sp(2))}(s, t) &= \frac{(1 - s^6 t^4)(1 - s^4 t^4)}{(1 - s^2)(1 - s^4)(1 - t^2)(1 - st^2)^3(1 - s^2 t^2)(1 - s^3 t^2)^3} , \\
&= 1 + s^2 + 2s^4 + 2s^6 + 3s^8 + t^2 + 3st^2 + 2s^2 t^2 + 6s^3 t^2 + 3s^4 t^2 + 9s^5 t^2 + 4s^6 t^2 + \\
&\quad 12s^7 t^2 + 5s^8 t^2 + t^4 + 3st^4 + 8s^2 t^4 + 9s^3 t^4 + 18s^4 t^4 + 15s^5 t^4 + 30s^6 t^4 + \\
&\quad 21s^7 t^4 + 40s^8 t^4 + t^6 + 3st^6 + 8s^2 t^6 + 19s^3 t^6 + 24s^4 t^6 + 43s^5 t^6 + 45s^6 t^6 + \\
&\quad 74s^7 t^6 + 66s^8 t^6 + t^8 + 3st^8 + 8s^2 t^8 + 19s^3 t^8 + 39s^4 t^8 + 53s^5 t^8 + 90s^6 t^8 + \\
&\quad 102s^7 t^8 + 156s^8 t^8 + O(s^9)O(t^9) , \\
g^{(2,Sp(2))}(s, t) &= 1 + s^2 + 2s^4 + 2s^6 + 3s^8 + 6t^2 + 10st^2 + 12s^2 t^2 + 20s^3 t^3 + 18s^4 t^2 + 30s^5 t^2 + \\
&\quad 24s^6 t^2 + 40s^7 t^2 + 30s^8 t^2 + 21t^4 + 60st^4 + 111s^2 t^4 + 165s^3 t^4 + 232s^4 t^4 + 270s^5 t^4 + \\
&\quad 357s^6 t^4 + 375s^7 t^4 + 478s^8 t^4 + 56t^6 + 210st^6 + 500s^2 t^6 + 890s^3 t^6 + 1330s^4 t^6 + \\
&\quad 1836s^5 t^6 + 2300s^6 t^6 + 2866s^7 t^6 + 3270s^8 t^6 + 126t^8 + 560st^8 + 1560s^2 t^8 + \\
&\quad 3220s^3 t^8 + 5405s^4 t^8 + 7991s^5 t^8 + 10955s^6 t^8 + 13906s^7 t^8 + 18190s^8 t^8 + O(s^9)O(t^9) , \\
g^{(3,Sp(2))}(s, t) &= 1 + s^2 + 2s^4 + 2s^6 + 3s^8 + 15t^2 + 21st^2 + 30s^2 t^2 + 42s^3 t^2 + 45s^4 t^2 + 63s^5 t^2 + \\
&\quad 60s^6 t^2 + 84s^7 t^2 + 75s^8 t^2 + 120t^4 + 315st^4 + 561s^2 t^4 + 840s^3 t^4 + 1122s^4 t^4 + \\
&\quad 1365s^5 t^4 + 1689s^6 t^4 + 1890s^7 t^4 + 2250s^8 t^4 + 679t^6 + 2485st^6 + 5530s^2 t^6 + \\
&\quad 9436s^3 t^6 + 13895s^4 t^6 + 18571s^5 t^6 + 23310s^6 t^6 + 28168s^7 t^6 + 32725s^8 t^6 + \\
&\quad 3045t^8 + 13770st^8 + 36120s^2 t^8 + 70050s^3 t^8 + 113190s^4 t^8 + 162960s^5 t^8 + \\
&\quad 216825s^6 t^8 + 272160s^7 t^8 + 328896s^8 t^8 + O(s^9)O(t^9) , \\
g^{(4,Sp(2))}(s, t) &= 1 + s^2 + 2s^4 + 2s^6 + 3s^8 + 28t^2 + 36st^2 + 56s^2 t^2 + 72s^3 t^2 + 84s^4 t^2 + 108s^5 t^2 + \\
&\quad 112s^6 t^2 + 144s^7 t^2 + 140s^8 t^2 + 406t^4 + 1008st^4 + 1786s^2 t^4 + 2646s^3 t^4 + 3488s^4 t^4 + \\
&\quad 4284s^5 t^4 + 5198s^6 t^4 + 5922s^7 t^4 + 6900s^8 t^4 + 4032t^6 + 14196st^6 + 30576s^2 t^6 + \\
&\quad 51192s^3 t^6 + 74424s^4 t^6 + 98352s^5 t^6 + 122892s^6 t^6 + 147228s^7 t^6 + 171360s^8 t^6 + \\
&\quad 30744t^8 + 135120st^8 + 340641s^2 t^8 + 640800s^3 t^8 + 1015092s^4 t^8 + 1438332s^5 t^8 + \\
&\quad 1889421s^6 t^8 + 2351496s^7 t^8 + 2819127s^8 t^8 + O(s^9)(t^9) . \tag{5.2}
\end{aligned}$$

A general form of the generating function when we set $s = t = \tilde{t}$ is

$$g^{(N_f, Sp(2))}(t) = \frac{P_{24N_f-16}(t)}{(1+t)^{6N_f-7}(1-t^2)^{2N_f+1}(1+t^2)(1-t^3)^{4N_f-1}(1-t^5)^{2N_f}} . \tag{5.3}$$

Plethystic Logarithms. We shall calculate the plethystic logarithms of the gener-

ating functions using formula (2.16):

$$\begin{aligned}
\text{PL} \left[g^{(1,Sp(2))}(s, t) \right] &= s^2 + s^4 + t^2 + 3st^2 + s^2t^2 + 3s^3t^2 - s^4t^4 - s^6t^4, \\
\text{PL} \left[g^{(2,Sp(2))}(s, t) \right] &= s^2 + s^4 + 6t^2 + 10st^2 + 6s^2t^2 + 10s^3t^2 - s^2t^4 - 15s^3t^4 - 21s^4t^4 - 15s^5t^4 \\
&\quad - 20s^6t^4 - 6s^2t^6 - 10s^3t^6 + 16s^5t^6 + 86s^6t^6 + 90s^7t^6 + 80s^8t^6 + 15s^3t^8 \\
&\quad + 51s^4t^8 + 140s^5t^8 + 120s^6t^8 - 31s^7t^8 - 236s^8t^8 + O(s^9)O(t^9), \\
\text{PL} \left[g^{(3,Sp(2))}(s, t) \right] &= s^2 + s^4 + 15t^2 + 21st^2 + 15s^2t^2 + 21s^3t^2 - 15s^2t^4 - 105s^3t^4 - 120s^4t^4 \\
&\quad - 105s^5t^4 - 105s^6t^4 - t^6 - 35st^6 - 189s^2t^6 - 175s^3t^6 + 36s^4t^6 + 539s^5t^6 \\
&\quad + 1589s^6t^6 + 1575s^7t^6 + 1365s^8t^6 + 36st^8 + 540s^2t^8 + 2100s^3t^8 + 5019s^4t^8 \\
&\quad + 8118s^5t^8 + 4185s^6t^8 - 4515s^7t^8 - 16869s^8t^8 + O(s^9)O(t^9), \\
\text{PL} \left[g^{(4,Sp(2))}(s, t) \right] &= s^2 + s^4 + 28t^2 + 36st^2 + 28s^2t^2 + 36s^3t^2 - 70s^2t^4 - 378s^3t^4 - 406s^4t^4 \\
&\quad - 378s^5t^4 - 336s^6t^4 - 28t^6 - 420st^6 - 1512s^2t^6 - 1176s^3t^6 + 448s^4t^6 \\
&\quad + 4452s^5t^6 + 10920s^6t^6 + 10584s^7t^6 + 8988s^8t^6 + 63t^8 + 1728st^8 \\
&\quad + 12481s^2t^8 + 38220s^3t^8 + 80493s^4t^8 + 108432s^5t^8 + 41615s^6t^8 \\
&\quad - 81564s^7t^8 - 247521s^8t^8 + O(s^9)O(t^9). \tag{5.4}
\end{aligned}$$

5.1.2 The $Sp(3)$ Gauge Group

We now move to examining the generating functions and their plethystic logarithms for the $Sp(3)$ gauge group with $2N_f$ chiral fields transforming in the fundamental representation and one in the adjoint representation of the group. The Molien-Weyl formula for this theory is

$$\begin{aligned}
g^{(N_f, Sp(3))} &= \int_{Sp(3)} d\mu_{Sp(3)}(z_1, z_2, z_3) \text{PE} [2N_f[1, 0, 0]t + [2, 0, 0]s] \\
&= \oint_{|z_1|=1} \frac{dz_1}{2\pi iz_1} \oint_{|z_2|=1} \frac{dz_2}{2\pi iz_2} \oint_{|z_3|=1} \frac{dz_3}{2\pi iz_3} \frac{(1 - z_1^2)(1 - \frac{z_1^2}{z_2})(1 - z_2)(1 - \frac{z_2^2}{z_1})(1 - \frac{z_1 z_2}{z_3})}{(1 - s\frac{1}{z_1})(1 - sz_1^2)(1 - s\frac{z_1^2}{z_2})(1 - s\frac{1}{z_2})(1 - s\frac{z_1^2}{z_2})} \times \\
&\quad \frac{(1 - \frac{z_2^2}{z_1 z_3})(1 - \frac{z_3}{z_1})(1 - \frac{z_1 z_3}{z_2})(1 - \frac{z_3^2}{z_2^2})}{(1 - s\frac{z_2^2}{z_1})(1 - s\frac{z_2^2}{z_3})(1 - s\frac{z_1}{z_3})(1 - s\frac{z_2}{z_1 z_3})(1 - s\frac{z_1 z_2}{z_3})(1 - s\frac{z_2^2}{z_1 z_3})(1 - s\frac{z_3}{z_1})(1 - s\frac{z_1 z_3}{z_2^2})} \times \\
&\quad \frac{1}{(1 - s\frac{z_3}{z_1 z_2})(1 - s\frac{z_2}{z_1^2})(1 - s\frac{z_1 z_3}{z_2})(1 - s\frac{z_3^2}{z_2^2})(1 - sz_2)(1 - s)^3((1 - t\frac{1}{z_1})(1 - tz_1))^{2N_f}} \times \\
&\quad \frac{1}{((1 - t\frac{z_1}{z_2})(1 - t\frac{z_2}{z_1})(1 - t\frac{z_2}{z_3})(1 - t\frac{z_3}{z_2}))^{2N_f}} \tag{5.1}
\end{aligned}$$

Applying the residue theorem we can compute Hilbert series for various N_f :

$$\begin{aligned}
g^{(1,Sp(3))}(s, t) &= \frac{(1 - s^6 t^4)(1 - s^8 t^4)(1 - s^{10} t^4)}{(1 - s^2)(1 - s^4)(1 - s^6)(1 - t^2)(1 - st^2)^3(1 - s^2 t^2)(1 - s^3 t^2)^3(1 - s^4 t^2)(1 - s^5 t^2)^3} , \\
&= 1 + s^2 + 2s^4 + 3s^6 + 4s^8 + t^2 + 3st^2 + 2s^2 t^2 + 6s^3 t^2 + 4s^4 t^2 + 12s^5 t^2 + 6s^6 t^2 + \\
&\quad 18s^7 t^2 + 9s^8 t^2 + t^4 + 3st^4 + 8s^2 t^4 + 9s^3 t^4 + 20s^4 t^4 + 21s^5 t^4 + 43s^6 t^4 + \\
&\quad 36s^7 t^4 + 71s^8 t^4 + t^6 + 3st^6 + 8s^2 t^6 + 19s^3 t^6 + 26s^4 t^6 + 52s^5 t^6 + 65s^6 t^6 + \\
&\quad 116s^7 t^6 + 123s^8 t^6 + t^8 + 3st^8 + 8s^2 t^8 + 19s^3 t^8 + 41s^4 t^8 + 62s^5 t^8 + 116s^6 t^8 + \\
&\quad 157s^7 t^8 + 265s^8 t^8 + O(s^9)O(t^9) , \\
g^{(2,Sp(3))}(s, t) &= 1 + s^2 + 2s^4 + 3s^6 + 4s^8 + 6t^2 + 10st^2 + 12s^2 t^2 + 20s^3 t^2 + 24s^4 t^2 + 40s^5 t^2 + \\
&\quad 36s^6 t^2 + 60s^7 t^2 + 54s^8 t^2 + 21t^4 + 60st^4 + 112s^2 t^4 + 180s^3 t^4 + 289s^4 t^4 + 405s^5 t^4 + \\
&\quad 571s^6 t^4 + 690s^7 t^4 + 939s^8 t^4 + 56t^6 + 210st^6 + 512s^2 t^6 + 1000s^3 t^6 + 1738s^4 t^6 + \\
&\quad 2790s^5 t^6 + 4094s^6 t^6 + 5770s^7 t^6 + 7600s^8 t^6 + 126t^8 + 560st^8 + 1617s^2 t^8 + 3700s^3 t^8 + \\
&\quad 7257s^4 t^8 + 12725s^5 t^8 + 20552s^6 t^8 + 30860s^7 t^8 + 44162s^8 t^8 + O(s^9)O(t^9) , \\
g^{(3,Sp(3))}(s, t) &= 1 + s^2 + 2s^4 + 3s^6 + 4s^8 + 15t^2 + 21st^2 + 30s^2 t^2 + 42s^3 t^2 + 60s^4 t^2 + 84s^5 t^2 + \\
&\quad 90s^6 t^2 + 126s^7 t^2 + 135s^8 t^2 + 120t^4 + 315st^4 + 576s^2 t^4 + 945s^3 t^4 + 1467s^4 t^4 + \\
&\quad 2100s^5 t^4 + 2820s^6 t^4 + 3570s^7 t^4 + 4608s^8 t^4 + 680t^6 + 2520st^6 + 5944s^2 t^6 + 11501s^3 t^6 + \\
&\quad 19889s^4 t^6 + 31262s^5 t^6 + 45629s^6 t^6 + 63000s^7 t^6 + 83549s^8 t^6 + 3060t^8 + 14280st^8 + \\
&\quad 40950s^2 t^8 + 92274s^3 t^8 + 178626s^4 t^8 + 308589s^5 t^8 + 488922s^6 t^8 + 724509s^7 t^8 + \\
&\quad 1019424s^8 t^8 + O(s^9)O(t^9) , \\
g^{(4,Sp(3))}(s, t) &= 1 + s^2 + 2s^4 + 3s^6 + 4s^8 + 28t^2 + 36st^2 + 56s^2 t^2 + 72s^3 t^2 + 112s^4 t^2 + 144s^5 t^2 + \\
&\quad 168s^6 t^2 + 216s^7 t^2 + 252s^8 t^2 + 406t^4 + 1008st^4 + 1856s^2 t^4 + 3024s^3 t^4 + 4678s^4 t^4 + \\
&\quad 6678s^5 t^4 + 8874s^6 t^4 + 11340s^7 t^4 + 14442s^8 t^4 + 4060t^6 + 14616st^6 + 34048s^2 t^6 + \\
&\quad 65472s^3 t^6 + 112280s^4 t^6 + 175044s^5 t^6 + 253764s^6 t^6 + 348276s^7 t^6 + 460768s^8 t^6 + \\
&\quad 31464t^8 + 146097st^8 + 414036s^2 t^8 + 922593s^3 t^8 + 1765443s^4 t^8 + 3014997s^5 t^8 + \\
&\quad 4727424s^6 t^8 + 6942129s^7 t^8 + 9691512s^8 t^8 + O(s^9)(t^9) . \tag{5.2}
\end{aligned}$$

Plethystic Logarithms. We shall calculate the plethystic logarithms of the gener-

ating functions using formula (2.16):

$$\begin{aligned}
\text{PL} \left[g^{(1, Sp(3))}(s, t) \right] &= s^2 + s^4 + s^6 + t^2 + 3st^2 + s^2t^2 + 3s^3t^2 + s^4t^2 + 3s^5t^2 - s^6t^4 - s^8t^4 - s^{10}t^4, \\
\text{PL} \left[g^{(2, Sp(3))}(s, t) \right] &= s^2 + s^4 + s^6 + 6t^2 + 10st^2 + 6s^2t^2 + 10s^3t^2 + 6s^4t^2 + 10s^5t^2 - s^4t^4 - 15s^5t^4 - \\
&\quad 21s^6t^4 - 15s^7t^4 - 21s^8t^4 - 6s^4t^6 - 10s^5t^6 - 6s^6t^6 - 10s^7t^6 - s^4t^8 + 15s^7t^8 + \\
&\quad 52s^8t^8 + O(s^9)O(t^9), \\
\text{PL} \left[g^{(3, Sp(3))}(s, t) \right] &= s^2 + s^4 + s^6 + 15t^2 + 21st^2 + 15s^2t^2 + 21s^3t^2 + 15s^4t^2 + 21s^5t^2 - 15s^4t^4 - \\
&\quad 105s^5t^4 - 120s^6t^4 - 105s^7t^4 - 120s^8t^4 - s^2t^6 - 35s^3t^6 - 190s^4t^6 - 210s^5t^6 - \\
&\quad 189s^6t^6 - 175s^7t^6 + 36s^8t^6 - 15s^2t^8 - 105s^3t^8 - 105s^4t^8 + 36s^5t^8 + 555s^6t^8 + \\
&\quad 2241s^7t^8 + 5679s^8t^8 + O(s^9)O(t^9), \\
\text{PL} \left[g^{(4, Sp(3))}(s, t) \right] &= s^2 + s^4 + s^6 + 28t^2 + 36st^2 + 28s^2t^2 + 36s^3t^2 + 28s^4t^2 + 36s^5t^2 - 70s^4t^4 - \\
&\quad 378s^5t^4 - 406s^6t^4 - 378s^7t^4 - 406s^8t^4 - 28s^2t^6 - 420s^3t^6 - 1540s^4t^6 - 1596s^5t^6 - \\
&\quad 1512s^6t^6 - 1176s^7t^6 + 448s^8t^6 - t^8 - 63st^8 - 720s^2t^8 - 2352s^3t^8 - 1700s^4t^8 + \\
&\quad 1791s^5t^8 + 13265s^6t^8 + 42363s^7t^8 + 95521s^8t^8 + O(s^9)O(t^9). \tag{5.3}
\end{aligned}$$

5.2 Generators of the Chiral Ring

Below we summarise the generators of $Sp(N_c)$ adjoint SQCD [23, 24] and representations of $SU(2N_f) \times SU(2N_f)$ in which they transform.

$$\begin{aligned}
\text{Casimir invariants : } s^{2k} &\rightarrow u_{2k} = \text{Tr}(\phi^{2k}) \quad (k = 1, 2, \dots, N_c) \\
&\quad [0, \dots, 0] \quad 1 \text{ dimensional}, \\
\text{Mesons : } t^2 &\rightarrow M^{ij} = J^{ab} Q_a^i Q_b^j \quad (a, b = 1, 2, \dots, 2N_c) \\
&\quad [0, 1, 0, \dots, 0] \quad N_f(2N_f - 1) \text{ dimensional}, \\
\text{Even adjoint mesons : } s^{2l}t^2 &\rightarrow (A_{2l})^{ij} = J^{a_1 b_1} J^{c_1 b_2} J^{c_2 b_3} \dots J^{c_{2l-1} b_{2l}} J^{c_{2l} a_{2l}} Q_{a_1}^i \phi_{b_1 c_1} \phi_{b_2 c_2} \dots \phi_{b_{2l} c_{2l}} Q_{a_{2l}}^j \\
&\quad [0, 1, 0, \dots, 0] \quad N_f(2N_f - 1) \text{ dimensional} \quad (l = 1, \dots, N_c - 1), \\
\text{Odd adjoint mesons : } s^{2k-1}t^2 &\rightarrow (A_{2k-1})^{ij} = J^{a_1 b_1} J^{c_1 b_2} \dots J^{c_{2k-1} a_{2k-1}} Q_{a_1}^i \phi_{b_1 c_1} \phi_{b_2 c_2} \dots \phi_{b_{2k-1} c_{2k-1}} Q_{a_{2k-1}}^j \\
&\quad [2, 0, \dots, 0] \quad N_f(2N_f + 1) \text{ dimensional} \quad (k = 1, \dots, N_c),
\end{aligned}$$

The total number of generators is

$$N_c(1 + 4N_f^2). \tag{5.4}$$

We note that for $N_c = 1$, $Sp(1)$ is isomorphic to $SU(2)$. In which case, we recover the $SU(2)$ adjoint SQCD and hence (5.4) reduces to (4.7).

5.3 Complete Intersection Moduli Space

We claim that the moduli space of the $Sp(N_c)$ adjoint SQCD with 2 fundamental chiral multiplets ($N_f = 1$) and 1 adjoint chiral multiplet is a complete intersection. A general expression of the phethystic logarithm for the complete intersection case can be written as

$$\text{PL}[g^{(1,Sp(N_c))}(s, t)] = \sum_{k=1}^{N_c} s^{2k} + \sum_{k=1}^{N_c} (3s^{2k-1} + s^{2(k-1)})t^2 - \sum_{k=1}^{N_c} s^{2(N_c+k-1)}t^4. \quad (5.5)$$

Note that the number of relations is equal to the rank of the gauge group and not to 1 as might naively be expected from the case of $SU(N_c)$ gauge group.

6. The $SO(N_c)$ Gauge Groups

Let us turn to the $SO(N_c)$ gauge theory with N_f chiral superfields transforming in the fundamental (vector) representation (N_f flavours) and 1 chiral superfield transforming in the adjoint representation. The global symmetry of this theory [23] is $SU(N_f) \times U(1)_R$.

6.1 Examples of Hilbert Series

We shall derive Hilbert series for various cases. Note that the following subsections are *not* merely a collection of results. They will turn out to be essential for analysing the generators of the chiral ring.

6.1.1 The $SO(3)$ Gauge Group

Let us now examine the $SO(3)$ gauge theory with N_f chiral multiplets in the fundamental representation and 1 chiral superfield in the adjoint representation. The Molien–Weyl formula for this theory is

$$\begin{aligned} g^{(N_f, SO(3))} &= \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} (1-z) \text{PE} [N_f[1]t + [1]s] \\ &= \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} \frac{1-z}{(1-t)^{N_f}(1-tz)^{N_f}(1-\frac{t}{z})^{N_f}(1-s)(1-sz)(1-\frac{s}{z})}. \end{aligned} \quad (6.1)$$

Applying the residue theorem, we find that

$$\begin{aligned}
g^{(1,SO(3))}(s, t) &= \frac{1}{(1-s^2)(1-st)(1-t^2)} , \\
g^{(2,SO(3))}(s, t) &= \frac{1-s^2t^4}{(1-s^2)(1-st)^2(1-st^2)(1-t^2)^3} , \\
g^{(3,SO(3))}(s, t) &= \frac{1-t+t^2+3st^2-3st^3-s^2t^3+s^2t^4-s^2t^5}{(1-s^2)(1-t)^6(1+t)^5(1-st)^3} , \\
g^{(4,SO(3))}(s, t) &= 1 + s^2 + s^4 + 4st + 4s^3t + 10t^2 + 6st^2 + 20s^2t^2 + 6s^3t^2 + 20s^4t^2 + 4t^3 \\
&\quad + 40st^3 + 24s^2t^3 + 60s^3t^3 + 24s^4t^3 + 55t^4 + 60st^4 + 155s^2t^4 + 105s^3t^4 \\
&\quad + 190s^4t^4 + O(s^5)O(t^5) . \tag{6.2}
\end{aligned}$$

Plethystic logarithms. We shall calculate plethystic logarithms of generating functions using formula (2.16):

$$\begin{aligned}
\text{PL} [g^{(1,SO(3))}(s, t)] &= s^2 + st + t^2 , \\
\text{PL} [g^{(2,SO(3))}(s, t)] &= s^2 + 2st + 3t^2 + st^2 - s^2t^4 , \\
\text{PL} [g^{(3,SO(3))}(s, t)] &= s^2 + 3st + 6t^2 + 3st^2 + t^3 - s^2t^3 - 3st^4 - 6s^2t^4 + O(s^5)O(t^5) , \\
\text{PL} [g^{(4,SO(3))}(s, t)] &= s^2 + 4st + 10t^2 + 6st^2 + 4t^3 - 4s^2t^3 - 16st^4 - 21s^2t^4 + s^3t^4 + O(s^5)O(t^5) . \tag{6.3}
\end{aligned}$$

The $N_f = 1$ moduli space is freely generated, *i.e.* there is no relation between the generators. Since the plethystic logarithm for $N_f = 2$ is a polynomial (not an infinite series), it follows that the $N_f = 2$ moduli space is a complete intersection.

6.1.2 The $SO(4)$ Gauge Group

Let us now examine the $SO(4)$ gauge theory with N_f chiral multiplets in the fundamental representation and 1 chiral superfield in the adjoint representation. The Molien–Weyl formula for this theory is

$$\begin{aligned}
g^{(N_f, SO(4))} &= \frac{1}{(2\pi i)^2} \oint_{|z_1|=1} \frac{dz_1}{z_1} \oint_{|z_2|=1} \frac{dz_2}{z_2} \left(1 - \frac{z_1}{z_2}\right) (1 - z_1 z_2) \\
&\quad \times \text{PE} [N_f[1, 0]t + [1, 1]s] . \tag{6.4}
\end{aligned}$$

Applying the residue theorem, we find that

$$\begin{aligned}
g^{(1,SO(4))}(s,t) &= \frac{1}{(1-s^2)^2(1-t^2)(1-s^2t^2)} , \\
g^{(2,SO(4))}(s,t) &= \frac{(1-s^2t^4)(1-s^4t^4)}{(1-s^2)^2(1-t^2)^3(1-st^2)^2(1-s^2t^2)^3} , \\
g^{(3,SO(4))}(s,t) &= \frac{1+3st^2+3s^2t^2+3s^3t^4-2s^2t^6-8s^3t^6-8s^4t^6-2s^5t^6+3s^4t^8+3s^5t^{10}+3s^6t^{10}+s^7t^{12}}{(1-s^2)^2(1-st)^3(1+st)^3(1-t^2)^6(1-st^2)^3} , \\
g^{(4,SO(4))}(s,t) &= 1+2s^2+3s^4+10t^2+12st^2+30s^2t^2+24s^3t^2+50s^4t^2+56t^4+120st^4+267s^2t^4+ \\
&\quad 330s^3t^4+513s^4t^4+O(s^5)O(t^5) , \\
g^{(5,SO(4))}(s,t) &= 1+2s^2+3s^4+15t^2+20st^2+45s^2t^2+40s^3t^2+75s^4t^2+125t^4+300st^4+620s^2t^4+ \\
&\quad 810s^3t^4+1185s^4t^4+O(s^5)O(t^5) .
\end{aligned} \tag{6.5}$$

Plethystic logarithms. We shall calculate plethystic logarithms of generating functions using formula (2.16):

$$\begin{aligned}
\text{PL} [g^{(1,SO(4))}(s,t)] &= 2s^2 + t^2 + s^2t^2 , \\
\text{PL} [g^{(2,SO(4))}(s,t)] &= 2s^2 + 3t^2 + 2st^2 + 3s^2t^2 - s^2t^4 - s^4t^4 , \\
\text{PL} [g^{(3,SO(4))}(s,t)] &= 2s^2 + 6t^2 + 6st^2 + 6s^2t^2 - 6s^2t^4 - 6s^3t^4 - 6s^4t^4 + O(s^5)O(t^5) , \\
\text{PL} [g^{(4,SO(4))}(s,t)] &= 2s^2 + 10t^2 + 12st^2 + 10s^2t^2 + t^4 - 23s^2t^4 - 30s^3t^4 - 20s^4t^4 + O(s^5)O(t^5) , \\
\text{PL} [g^{(5,SO(4))}(s,t)] &= 2s^2 + 15t^2 + 20st^2 + 15s^2t^2 + 5t^4 - 65s^2t^4 - 90s^3t^4 - 50s^4t^4 + O(s^5)O(t^5) , \\
\text{PL} [g^{(6,SO(4))}(s,t)] &= 2s^2 + 21t^2 + 30st^2 + 21s^2t^2 + 15t^4 - 150s^2t^4 - 210s^3t^4 - 105s^4t^4 + O(s^5)O(t^5) .
\end{aligned} \tag{6.6}$$

The $N_f = 1$ moduli space is freely generated, *i.e.* there is no relation between the generators. Since the plethystic logarithm for $N_f = 2$ is a polynomial (not an infinite series), it follows that the $N_f = 2$ moduli space is a complete intersection.

6.1.3 The $SO(5)$ Gauge Group

Let us now examine the $SO(5)$ gauge theory with N_f chiral multiplets in the fundamental representation $[1, 0]$ and 1 chiral superfield in the adjoint representation $[0, 2]$. The Molien–Weyl formula for this theory is

$$\begin{aligned}
g^{(N_f,SO(5))} &= \frac{1}{(2\pi i)^2} \oint_{|z_1|=1} \frac{dz_1}{z_1} \oint_{|z_2|=1} \frac{dz_2}{z_2} (1-z_1) \left(1 - \frac{z_1^2}{z_2^2}\right) (1-z_2^2) \left(1 - \frac{z_2^2}{z_1}\right) \\
&\quad \times \text{PE} [N_f[1, 0]t + [0, 2]s] .
\end{aligned} \tag{6.7}$$

Applying the residue theorem, we find that

$$\begin{aligned}
g^{(1,SO(5))}(s, t) &= \frac{1}{(1-s^2)(1-s^4)(1-s^2t)(1-t^2)(1-s^2t^2)} , \\
g^{(2,SO(5))}(s, t) &= \frac{(1-s^4t^4)(1-s^6t^4)}{(1-s^2)(1-s^4)(1-s^2t)^2(1-t^2)^3(1-st^2)(1-s^2t^2)^3(1-s^3t^2)} , \\
g^{(3,SO(5))}(s, t) &= \frac{1}{(1-s)^2(1+s)^2(1+s^2)(1-t)^6(1+t)^6(1-st)^3(1+st)^3(1-s^2t)^3(1-st^2)^3} \times \\
&\quad (1+3s^2t^2+3s^3t^2+st^3-s^5t^3-3s^3t^4+6s^5t^4+3s^3t^5-3s^7t^5-s^4t^6-9s^5t^6-9s^6t^6 \\
&\quad -s^7t^6-3s^4t^7+3s^8t^7+6s^6t^8-3s^8t^8-s^6t^9+s^{10}t^9+3s^8t^{10}+3s^9t^{10}+s^{11}t^{12}) .
\end{aligned} \tag{6.8}$$

Plethystic logarithms. We shall calculate plethystic logarithms of generating functions using formula (2.16):

$$\begin{aligned}
\text{PL} [g^{(1,SO(5))}(s, t)] &= s^2 + s^4 + s^2t + t^2 + s^2t^2 , \\
\text{PL} [g^{(2,SO(5))}(s, t)] &= s^2 + s^4 + 2s^2t + 3t^2 + st^2 + 3s^2t^2 + s^3t^2 - s^4t^4 - s^6t^4 , \\
\text{PL} [g^{(3,SO(5))}(s, t)] &= s^2 + s^4 + 3s^2t + 6t^2 + 3st^2 + 6s^2t^2 + 3s^3t^2 + st^3 - s^5t^3 - 3s^3t^4 \\
&\quad - 6s^4t^4 - 3s^5t^4 - 6s^6t^4 - 3s^4t^5 - s^2t^6 - s^4t^6 + 9s^6t^6 + O(s^7)O(t^7) , \\
\text{PL} [g^{(4,SO(5))}(s, t)] &= s^2 + s^4 + 4s^2t + 10t^2 + 6st^2 + 10s^2t^2 + 6s^3t^2 + 4st^3 - 4s^5t^3 - 16s^3t^4 \\
&\quad - 21s^4t^4 - 15s^5t^4 - 21s^6t^4 - 4s^3t^5 - 24s^4t^5 - 10s^2t^6 - 6s^3t^6 - 10s^4t^6 \\
&\quad + 10s^5t^6 + 102s^6t^6 + O(s^7)O(t^7) , \\
\text{PL} [g^{(5,SO(5))}(s, t)] &= s^2 + s^4 + 5s^2t + 15t^2 + 10st^2 + 15s^2t^2 + 10s^3t^2 + 10st^3 - 10s^5t^3 \\
&\quad - 50s^3t^4 - 55s^4t^4 - 45s^5t^4 - 55s^6t^4 + t^5 - s^2t^5 - 24s^3t^5 - 101s^4t^5 + s^5t^5 \\
&\quad + s^6t^5 - 60s^2t^6 - 45s^3t^6 - 50s^4t^6 + 75s^5t^6 + 55s^6t^6 + O(s^7)O(t^7) .
\end{aligned} \tag{6.9}$$

The $N_f = 1$ moduli space is freely generated, *i.e.* there is no relation between the generators. Since the plethystic logarithm for $N_f = 2$ is a polynomial (not an infinite series), it follows that the $N_f = 2$ moduli space is a complete intersection.

6.1.4 The $SO(6)$ Gauge Group

Let us now examine the $SO(6)$ gauge theory with N_f chiral multiplets in the fundamental representation and 1 chiral superfield in the adjoint representation. The

Molien–Weyl formula for this theory is

$$g^{(N_f, SO(6))} = \frac{1}{(2\pi i)^3} \oint_{|z_1|=1} \frac{dz_1}{z_1} \oint_{|z_2|=1} \frac{dz_2}{z_2} \oint_{|z_3|=1} \frac{dz_3}{z_3} \left(1 - \frac{z_2^2}{z_1}\right) \left(1 - \frac{z_1^2}{z_2 z_3}\right) \left(1 - \frac{z_1 z_2}{z_3}\right) \times \\ \left(1 - \frac{z_1 z_3}{z_2}\right) (1 - z_2 z_3) \left(1 - \frac{z_3^2}{z_1}\right) \text{PE} [N_f[1, 0, 0]t + [0, 1, 1]s] . \quad (6.10)$$

Applying the residue theorem, we find that

$$\begin{aligned} g^{(1, SO(6))}(s, t) &= \frac{1}{(1-s^2)(1-s^3)(1-s^4)(1-t^2)(1-s^2 t^2)(1-s^4 t^2)} , \\ g^{(2, SO(6))}(s, t) &= \frac{(1-s^4 t^4)(1-s^6 t^4)(1-s^8 t^4)}{(1-s^2)(1-s^3)(1-s^4)(1-t^2)^3(1-st^2)(1-s^2 t^2)^4(1-s^3 t^2)(1-s^4 t^2)^3} , \\ g^{(3, SO(6))}(s, t) &= 1 + s^2 + s^3 + 2s^4 + s^5 + 3s^6 + 2s^7 + 4s^8 + 3s^9 + 5s^{10} + 6t^2 + 3st^2 + 15s^2 t^2 \\ &\quad + 12s^3 t^2 + 30s^4 t^2 + 24s^5 t^2 + 48s^6 t^2 + 42s^7 t^2 + 72s^8 t^2 + 63s^9 t^2 + 99s^{10} t^2 + 21t^4 \\ &\quad + 18st^4 + 81s^2 t^4 + 84s^3 t^4 + 204s^4 t^4 + 204s^5 t^4 + 381s^6 t^4 + 387s^7 t^4 + 621s^8 t^4 \\ &\quad + 624s^9 t^4 + 915s^{10} t^4 + 56t^6 + 63st^6 + 281s^2 t^6 + 354s^3 t^6 + 867s^4 t^6 + 1028s^5 t^6 \\ &\quad + 1907s^6 t^6 + 2182s^7 t^6 + 3446s^8 t^6 + 3825s^9 t^6 + 5474s^{10} t^6 + O(s^{11})O(t^7) , \\ g^{(4, SO(6))}(s, t) &= 1 + s^2 + s^3 + 2s^4 + s^5 + 3s^6 + 10t^2 + 6st^2 + 26s^2 t^2 + 22s^3 t^2 + 52s^4 t^2 \\ &\quad + 44s^5 t^2 + 84s^6 t^2 + 55t^4 + 61st^4 + 236s^2 t^4 + 272s^3 t^4 + 602s^4 t^4 + 654s^5 t^4 + \\ &\quad + 1139s^6 t^4 + 220t^6 + 340st^6 + 1316s^2 t^6 + 1916s^3 t^6 + 4252s^4 t^6 + 5540s^5 t^6 \\ &\quad + 9518s^6 t^6 + O(s^7)O(t^7) , \\ g^{(5, SO(6))}(s, t) &= 1 + s^2 + s^3 + 2s^4 + s^5 + 3s^6 + 15t^2 + 10st^2 + 40s^2 t^2 + 35s^3 t^2 + 80s^4 t^2 \\ &\quad + 70s^5 t^2 + 130s^6 t^2 + 120t^4 + 155st^4 + 550s^2 t^4 + 675s^3 t^4 + 1415s^4 t^4 + 1610s^5 t^4 \\ &\quad + 2695s^6 t^4 + 680t^6 + 1275st^6 + 4555s^2 t^6 + 7195s^3 t^6 + 15105s^4 t^6 + 20650s^5 t^6 \\ &\quad + 34055s^6 t^6 + O(s^7)O(t^7) , \\ g^{(5, SO(6))}(s, t) &= 1 + s^2 + s^3 + 2s^4 + s^5 + 3s^6 + 21t^2 + 15st^2 + 57s^2 t^2 + 51s^3 t^2 + 114s^4 t^2 \\ &\quad + 102s^5 t^2 + 186s^6 t^2 + 231t^4 + 330st^4 + 1107s^2 t^4 + 1416s^3 t^4 + 2865s^4 t^4 + 3357s^5 t^4 \\ &\quad + 5478s^6 t^4 + 1772t^6 + 3780st^6 + 12832s^2 t^6 + 21316s^3 t^6 + 43220s^4 t^6 + 60768s^5 t^6 \\ &\quad + 97744s^6 t^6 + O(s^7)O(t^7) . \end{aligned} \quad (6.11)$$

Plethystic logarithms. We shall calculate plethystic logarithms of generating func-

tions using formula (2.16):

$$\begin{aligned}
\text{PL} [g^{(1,SO(6))}(s, t)] &= s^2 + s^3 + s^4 + t^2 + s^2 t^2 + s^4 t^2 , \\
\text{PL} [g^{(2,SO(6))}(s, t)] &= s^2 + s^3 + s^4 + 3t^2 + st^2 + 4s^2 t^2 + s^3 t^2 + 3s^4 t^2 - s^4 t^4 - s^6 t^4 - s^8 t^4 , \\
\text{PL} [g^{(3,SO(6))}(s, t)] &= s^2 + s^3 + s^4 + 6t^2 + 3st^2 + 9s^2 t^2 + 3s^3 t^2 + 6s^4 t^2 - 6s^4 t^4 - 3s^5 t^4 \\
&\quad - 9s^6 t^4 - 3s^7 t^4 - 6s^8 t^4 - s^4 t^6 - s^5 t^6 - s^6 t^6 + 9s^8 t^6 + 9s^9 t^6 \\
&\quad + 17s^{10} t^6 + O(s^{11})O(t^7) , \\
\text{PL} [g^{(4,SO(6))}(s, t)] &= s^2 + s^3 + s^4 + 10t^2 + 6st^2 + 16s^2 t^2 + 6s^3 t^2 + 10s^4 t^2 + st^4 - 22s^4 t^4 \\
&\quad - 16s^5 t^4 - 36s^6 t^4 - 6s^3 t^6 - 16s^4 t^6 - 32s^5 t^6 - 10s^6 t^6 + O(s^7)O(t^7) , \\
\text{PL} [g^{(5,SO(6))}(s, t)] &= s^2 + s^3 + s^4 + 15t^2 + 10st^2 + 25s^2 t^2 + 10s^3 t^2 + 15s^4 t^2 + 5st^4 - 60s^4 t^4 \\
&\quad - 50s^5 t^4 - 100s^6 t^4 - 55s^3 t^6 - 105s^4 t^6 - 220s^5 t^6 - 45s^6 t^6 + O(s^7)O(t^7) , \\
\text{PL} [g^{(6,SO(6))}(s, t)] &= s^2 + s^3 + s^4 + 21t^2 + 15st^2 + 36s^2 t^2 + 15s^3 t^2 + 21s^4 t^2 + 15st^4 - 135s^4 t^4 \\
&\quad - 120s^5 t^4 - 225s^6 t^4 + t^6 - s^2 t^6 - 261s^3 t^6 - 436s^4 t^6 - 903s^5 t^6 - 139s^6 t^6 \\
&\quad + O(s^7)O(t^7) . \tag{6.12}
\end{aligned}$$

The $N_f = 1$ moduli space is freely generated, *i.e.* there is no relation between the generators. Since the plethystic logarithm for $N_f = 2$ is a polynomial (not an infinite series), it follows that the $N_f = 2$ moduli space is a complete intersection.

6.2 Generators of the Chiral Ring

Using plethystic logarithms computed in preceding subsections, we can write down the generators of $SO(N_c)$ adjoint SQCD and representations of $SU(N_f)$ in which they transform. We will make a distinction between $SO(2m)$ and $SO(2m+1)$ gauge groups.

6.2.1 The $SO(2m)$ Gauge Groups

The generators of the chiral ring in the case of $SO(2m)$ are as follows:

$$\begin{aligned}
\text{Casimir invariants : } s^{2k} &\rightarrow u_{2k} = \text{Tr}(\phi^{2k}) \quad k = 1, \dots, m-1 \\
&\quad [0, \dots, 0] \quad 1 \text{ dimensional} , \\
\text{Even adjoint mesons : } s^{2k} t^2 &\rightarrow (A_{2k})^{ij} = Q_{a_1}^i (\phi^{2k})_{a_1 a_2} Q_{a_2}^j \quad k = 0, \dots, m-1 \\
&\quad [2, 0, \dots, 0] \quad \frac{1}{2} N_f (N_f + 1) \text{ dimensional} , \\
\text{Odd adjoint mesons : } s^{2k+1} t^2 &\rightarrow (A_{2k+1})^{ij} = Q_{a_1}^i (\phi^{2k+1})_{a_1 a_2} Q_{a_2}^j \quad k = 0, \dots, m-2 \\
&\quad [0, 1, 0, \dots, 0] \quad \frac{1}{2} N_f (N_f - 1) \text{ dimensional} , \\
\text{Adjoint baryons : } s^k t^l &\rightarrow \mathcal{B}_k^{i_1 \dots i_l} = \epsilon^{a_1 \dots a_{2k} a_{2k+1} \dots a_{2k+l}} \phi_{a_1 a_2} \dots \phi_{a_{2k-1} a_{2k}} Q_{a_{2k+1}}^{i_1} \dots Q_{a_{2k+l}}^{i_l} \\
&\quad \text{with } 2k + l = 2m, \quad k = 0, \dots, m \\
&\quad [0, \dots, 0, 1_{l,L}, 0, \dots, 0] \quad \binom{N_f}{2m-2k} \text{ dimensional} .
\end{aligned}$$

The total number of generators is

$$m(1 + N_f^2) - \frac{N_f(N_f - 1)}{2} - 1 + \sum_{k=0}^m \binom{N_f}{2m - 2k}, \quad (6.13)$$

which behaves as $N_f^{N_c}/N_c!$ for large values of N_c .

6.2.2 The $SO(2m + 1)$ Gauge Groups

The generators of the chiral ring in the case of $SO(2m + 1)$ are as follows:

$$\begin{aligned} \text{Casimir invariants : } s^{2k} &\rightarrow u_{2k} = \text{Tr}(\phi^{2k}) \quad k = 1, \dots, m \\ &\quad [0, \dots, 0] \quad 1 \text{ dimensional}, \\ \text{Even adjoint mesons : } s^{2k} t^2 &\rightarrow (A_{2k})^{ij} = Q_{a_1}^i (\phi^{2k})_{a_1 a_2} Q_{a_2}^j \quad k = 0, \dots, m - 1 \\ &\quad [2, 0, \dots, 0] \quad \frac{1}{2} N_f (N_f + 1) \text{ dimensional}, \\ \text{Odd adjoint mesons : } s^{2k+1} t^2 &\rightarrow (A_{2k+1})^{ij} = Q_{a_1}^i (\phi^{2k+1})_{a_1 a_2} Q_{a_2}^j \quad k = 0, \dots, m - 1 \\ &\quad [0, 1, 0, \dots, 0] \quad \frac{1}{2} N_f (N_f - 1) \text{ dimensional}, \\ \text{Adjoint baryons : } s^k t^l &\rightarrow \mathcal{B}_k^{i_1 \dots i_l} = \epsilon^{a_1 \dots a_{2k} a_{2k+1} \dots a_{2k+l}} \phi_{a_1 a_2} \dots \phi_{a_{2k-1} a_{2k}} Q_{a_{2k+1}}^{i_1} \dots Q_{a_{2k+l}}^{i_l} \\ &\quad \text{with } 2k + l = 2m + 1, \quad k = 0, \dots, m \\ &\quad [0, \dots, 0, 1_{l;L}, 0, \dots, 0] \quad \binom{N_f}{2m+1-2k} \text{ dimensional} \end{aligned}$$

The total number of generators is

$$m(1 + N_f^2) + \sum_{k=0}^m \binom{N_f}{2m + 1 - 2k}, \quad (6.14)$$

which behaves as $N_f^{N_c}/N_c!$ for large values of N_c . It is to be noted that, among the adjoint mesons of $SO(N_c)$ gauge theories we have just listed, the generator A_0 is what we referred to as meson in the preceding sections. We have listed it among the adjoint mesons only for simplicity.

6.3 Complete Intersection Moduli Space

We have seen from several examples in the preceding sections that

- The moduli space of the $SO(N_c)$ gauge theories with 1 fundamental chiral superfield and 1 adjoint chiral superfield is freely generated,
- The moduli space of the $SO(N_c)$ gauge theories with 2 fundamental chiral superfields and 1 adjoint chiral superfield is a complete intersection.

Generalising these examples, we write down general expressions for the fully refined plethystic logarithms in the case of 2 fundamental chiral superfields as

$$\begin{aligned}
\text{PL}[g^{(2,SO(2m+1))}(s, t_1, t_2)] &= \sum_{k=1}^m s^{2k} + s^m t_1 + s^m t_2 + \sum_{k=1}^m [s^{2(k-1)}(t_1^2 + t_1 t_2 + t_2^2) + s^{2k-1} t_1 t_2] \\
&\quad - \sum_{k=1}^m s^{2(m+k-1)} (t_1 t_2)^2, \\
\text{PL}[g^{(2,SO(2m))}(s, t_1, t_2)] &= \sum_{k=1}^{m-1} s^{2k} + s^m + s^{m-1} t_1 t_2 + \sum_{k=1}^m s^{2(k-1)}(t_1^2 + t_1 t_2 + t_2^2) + \sum_{k=1}^{m-1} s^{2k-1} t_1 t_2 \\
&\quad - \sum_{k=1}^m s^{2(m+k-2)} (t_1 t_2)^2.
\end{aligned} \tag{6.15}$$

Note that the number of relations is equal to the rank of the gauge group and not to 1 as might naively be expected from the case of $SU(N_c)$ gauge group. The plethystic logarithms in the case of 1 fundamental chiral superfields can be easily obtained by setting $t_1 = t$, $t_2 = 0$:

$$\begin{aligned}
\text{PL}[g^{(1,SO(2m+1))}(s, t)] &= \sum_{k=1}^m s^{2k} + s^m t + \sum_{k=1}^m s^{2(k-1)} t^2, \\
\text{PL}[g^{(1,SO(2m))}(s, t)] &= \sum_{k=1}^{m-1} s^{2k} + s^m + \sum_{k=1}^m s^{2(k-1)} t^2.
\end{aligned} \tag{6.16}$$

7. The G_2 Gauge Group

Let us now examine the G_2 gauge theory with N_f chiral multiplets in the fundamental representation and 1 chiral superfield in the adjoint representation.

7.1 Examples of Hilbert Series

The Molien–Weyl formula for this theory is

$$\begin{aligned}
g^{(N_f, G_2)} &= \frac{1}{(2\pi i)^2} \oint_{|z_1|=1} \frac{dz_1}{z_1} \oint_{|z_2|=1} \frac{dz_2}{z_2} (1 - z_1) \left(1 - \frac{z_1^2}{z_2}\right) \left(1 - \frac{z_1^3}{z_2}\right) (1 - z_2) \times \\
&\quad \left(1 - \frac{z_2}{z_1}\right) \left(1 - \frac{z_2^2}{z_1^3}\right) \text{PE}[N_f[1, 0]t + [0, 1]s].
\end{aligned} \tag{7.1}$$

Applying the residue theorem, we find that

$$\begin{aligned}
g^{(1,G_2)}(s,t) &= \frac{1 - s^{12}t^6}{(1-s^2)(1-s^6)(1-s^3t)(1-t^2)(1-s^2t^2)(1-s^4t^2)(1-s^3t^3)(1-s^6t^3)} , \\
g^{(2,G_2)}(s,t) &= 1 + s^2 + s^4 + 2s^6 + 2s^8 + 2s^{10} + 3s^{12} + 2s^3t + 2s^5t + 2s^7t + 4s^9t + 4s^{11}t + \\
&\quad 3t^2 + st^2 + 6s^2t^2 + 2s^3t^2 + 9s^4t^2 + 3s^5t^2 + 15s^6t^2 + 4s^7t^2 + 18s^8t^2 + 5s^9t^2 + \\
&\quad 21s^{10}t^2 + 6s^{11}t^2 + 27s^{12}t^2 + 2st^3 + 2s^2t^3 + 12s^3t^3 + 6s^4t^3 + 20s^5t^3 + 12s^6t^3 + \\
&\quad 28s^7t^3 + 16s^8t^3 + 42s^9t^3 + 20s^{10}t^3 + 50s^{11}t^3 + 26s^{12}t^3 + 6t^4 + 3st^4 + 17s^2t^4 + \\
&\quad 12s^3t^4 + 37s^4t^4 + 25s^5t^4 + 70s^6t^4 + 41s^7t^4 + 99s^8t^4 + 61s^9t^4 + 128s^{10}t^4 + \\
&\quad 77s^{11}t^4 + 166s^{12}t^4 + 6st^5 + 8s^2t^5 + 38s^3t^5 + 32s^4t^5 + 84s^5t^5 + 74s^6t^5 + 146s^7t^5 + \\
&\quad 122s^8t^5 + 226s^9t^5 + 174s^{10}t^5 + 300s^{11}t^5 + 232s^{12}t^5 + 10t^6 + 6st^6 + 37s^2t^6 + \\
&\quad 36s^3t^6 + 105s^4t^6 + 100s^5t^6 + 233s^6t^6 + 198s^7t^6 + 385s^8t^6 + 330s^9t^6 + 560s^{10}t^6 + \\
&\quad 466s^{11}t^6 + 764s^{12}t^6 + O(s^{13})O(t^7) , \\
g^{(3,G_2)}(s,t) &= 1 + s^2 + s^4 + 2s^6 + 2s^8 + 2s^{10} + 3s^{12} + 3s^3t + 3s^5t + 3s^7t + 6s^9t + 6s^{11}t + 6t^2 + \\
&\quad 3st^2 + 12s^2t^2 + 6s^3t^2 + 18s^4t^2 + 9s^5t^2 + 30s^6t^2 + 12s^7t^2 + 36s^8t^2 + 15s^9t^2 + 42s^{10}t^2 + \\
&\quad 18s^{11}t^2 + 54s^{12}t^2 + t^3 + 8st^3 + 10s^2t^3 + 37s^3t^3 + 27s^4t^3 + 63s^5t^3 + 47s^6t^3 + 89s^7t^3 + \\
&\quad 64s^8t^3 + 128s^9t^3 + 81s^{10}t^3 + 154s^{11}t^3 + 101s^{12}t^3 + 21t^4 + 21st^4 + 72s^2t^4 + 75s^3t^4 + \\
&\quad 162s^4t^4 + 153s^5t^4 + 297s^6t^4 + 246s^7t^4 + 423s^8t^4 + 348s^9t^4 + 549s^{10}t^4 + 441s^{11}t^4 \\
&\quad + 699s^{12}t^4 + 6t^5 + 51st^5 + 90s^2t^5 + 255s^3t^5 + 300s^4t^5 + 570s^5t^5 + 621s^6t^5 + 978s^7t^5 \\
&\quad + 990s^8t^5 + 1464s^9t^5 + 1383s^{10}t^5 + 1932s^{11}t^5 + 1794s^{12}t^5 + 57t^6 + 89st^6 + 317s^2t^6 \\
&\quad + 471s^3t^6 + 980s^4t^6 + 1235s^5t^6 + 2104s^6t^6 + 2339s^7t^6 + 3487s^8t^6 + 3687s^9t^6 \\
&\quad + 5040s^{10}t^6 + 5106s^{11}t^6 + 6725s^{12}t^6 + O(s^{13})O(t^7) , \\
g^{(4,G_2)}(s,t) &= 1 + s^2 + s^4 + 2s^6 + 2s^8 + 2s^{10} + 3s^{12} + 4s^3t + 4s^5t + 4s^7t + 8s^9t + 8s^{11}t + \\
&\quad 10t^2 + 6st^2 + 20s^2t^2 + 12s^3t^2 + 30s^4t^2 + 18s^5t^2 + 50s^6t^2 + 24s^7t^2 + 60s^8t^2 + 30s^9t^2 + \\
&\quad 70s^{10}t^2 + 36s^{11}t^2 + 90s^{12}t^2 + 4t^3 + 20st^3 + 28s^2t^3 + 84s^3t^3 + 72s^4t^3 + 144s^5t^3 + \\
&\quad 120s^6t^3 + 204s^7t^3 + 164s^8t^3 + 288s^9t^3 + 208s^{10}t^3 + 348s^{11}t^3 + 256s^{12}t^3 + 56t^4 + \\
&\quad 75st^4 + 212s^2t^4 + 256s^3t^4 + 483s^4t^4 + 516s^5t^4 + 870s^6t^4 + 821s^7t^4 + 1241s^8t^4 \\
&\quad 1612s^{10}t^4 + 1447s^{11}t^4 + 2034s^{12}t^4 + 40t^5 + 224st^5 + 436s^2t^5 + 1028s^3t^5 + 1356s^4t^5 + \\
&\quad 2292s^5t^5 + 2696s^6t^5 + 3896s^7t^5 + 4232s^8t^5 + 5740s^9t^5 + 5852s^{10}t^5 + 7544s^{11}t^5 + \\
&\quad 7512s^{12}t^5 + 240t^6 + 550st^6 + 1640s^2t^6 + 2756s^3t^6 + 5140s^4t^6 + 7010s^5t^6 + 10820s^6t^6 + \\
&\quad 12980s^7t^6 + 17810s^8t^6 + 20036s^9t^6 + 25550s^{10}t^6 + 27466s^{11}t^6 + 33740s^{12}t^6 + \\
&\quad O(s^{13})O(t^7) .
\end{aligned} \tag{7.2}$$

Plethystic logarithms. We shall calculate plethystic logarithms of generating functions using formula (2.16):

$$\begin{aligned}
\text{PL} [g^{(1,G_2)}(s, t)] &= s^2 + s^6 + s^3t + t^2 + s^2t^2 + s^4t^2 + s^3t^3 + s^6t^3 - s^{12}t^6, \\
\text{PL} [g^{(2,G_2)}(s, t)] &= s^2 + s^6 + 2s^3t + 3t^2 + st^2 + 3s^2t^2 + s^3t^2 + 3s^4t^2 + s^5t^2 + 2st^3 + 2s^2t^3 \\
&\quad + 4s^3t^3 + 2s^4t^3 + 2s^5t^3 + 4s^6t^3 + s^2t^4 + 3s^3t^4 - s^6t^4 - 2s^8t^4 - 3s^9t^4 - s^{10}t^4 \\
&\quad - 4s^5t^5 - 8s^6t^5 - 8s^7t^5 - 8s^8t^5 - 8s^9t^5 - 8s^{10}t^5 - 4s^{11}t^5 - 3s^4t^6 - 5s^5t^6 \\
&\quad - 10s^6t^6 - 17s^7t^6 - 17s^8t^6 - 13s^9t^6 - 11s^{10}t^6 - 7s^{11}t^6 - 7s^{12}t^6 + O(s^{13})O(t^7), \\
\text{PL} [g^{(3,G_2)}(s, t)] &= s^2 + s^6 + 3s^3t + 6t^2 + 3st^2 + 6s^2t^2 + 3s^3t^2 + 6s^4t^2 + 3s^5t^2 + t^3 + 8st^3 + 9s^2t^3 \\
&\quad + 11s^3t^3 + 8s^4t^3 + 8s^5t^3 + 10s^6t^3 - s^8t^3 + 3st^4 + 9s^2t^4 + 15s^3t^4 - 3s^5t^4 - 6s^6t^4 \\
&\quad - 6s^7t^4 - 15s^8t^4 - 18s^9t^4 - 6s^{10}t^4 - 3s^3t^5 - 18s^4t^5 - 54s^5t^5 - 87s^6t^5 - 84s^7t^5 \\
&\quad - 81s^8t^5 - 78s^9t^5 - 63s^{10}t^5 - 27s^{11}t^5 + 6s^{12}t^5 - 10s^2t^6 - 35s^3t^6 - 81s^4t^6 \\
&\quad - 136s^5t^6 - 209s^6t^6 - 260s^7t^6 - 224s^8t^6 - 138s^9t^6 - 89s^{10}t^6 - 26s^{11}t^6 \\
&\quad + 33s^{12}t^6 + O(s^{13})O(t^7), \\
\text{PL} [g^{(4,G_2)}(s, t)] &= s^2 + s^6 + 4s^3t + 10t^2 + 6st^2 + 10s^2t^2 + 6s^3t^2 + 10s^4t^2 + 6s^5t^2 + 4t^3 + 20st^3 \\
&\quad + 24s^2t^3 + 24s^3t^3 + 20s^4t^3 + 20s^5t^3 + 20s^6t^3 - 4s^8t^3 + t^4 + 15st^4 + 35s^2t^4 + 45s^3t^4 \\
&\quad - 16s^5t^4 - 22s^6t^4 - 30s^7t^4 - 56s^8t^4 - 60s^9t^4 - 21s^{10}t^4 + s^{11}t^4 - 4s^2t^5 - 28s^3t^5 \\
&\quad - 124s^4t^5 - 288s^5t^5 - 424s^6t^5 - 408s^7t^5 - 380s^8t^5 - 356s^9t^5 - 260s^{10}t^5 - 96s^{11}t^5 \\
&\quad + 40s^{12}t^5 - 16st^6 - 116s^2t^6 - 332s^3t^6 - 646s^4t^6 - 1028s^5t^6 - 1454s^6t^6 - 1620s^7t^6 \\
&\quad - 1288s^8t^6 - 684s^9t^6 - 312s^{10}t^6 + 78s^{11}t^6 + 474s^{12}t^6 + O(s^{13})O(t^7). \tag{7.3}
\end{aligned}$$

7.2 Generators of the Chiral Ring

We note that G_2 has 3 independent invariant tensors [27, 28, 29]: δ^{ab} , $\epsilon^{a_1 \dots a_7}$ and $f^{a_1 a_2 a_3}$, where the last two tensors are totally antisymmetric and the indices run over 1 to 7. According to (2.10), we shall denote the adjoint field by an antisymmetric tensor ϕ^{ab} with the property

$$f_{abc}\phi^{ab} = 0. \tag{7.4}$$

Note that ϕ^{ab} has $\frac{1}{2}(7 \times 6) - 7 = 14$ independent components which is equal to the dimension of the adjoint representation of G_2 . Equipped with plethystic logarithms,

we can write down the generators of the chiral ring as follows:

$$\begin{aligned}
\text{Casimir invariants :} \quad & s^2 \rightarrow u_2 = \text{Tr}(\phi^2) \\
& [0, \dots, 0] \\
& s^6 \rightarrow u_6 = \text{Tr}(\phi^6) \\
& [0, \dots, 0] \\
\text{Adjoint-quark invariants :} \quad & s^3 t \rightarrow A^i = \epsilon^{a_1 \dots a_7} \phi_{a_1 a_2} \phi_{a_3 a_4} \phi_{a_5 a_6} Q_{a_7}^i \\
& [1, 0, \dots, 0] \\
\text{Even adjoint mesons :} \quad & s^{2k} t^2 \rightarrow (A_{2k})^{ij} = Q_{a_1}^i (\phi^{2k})_{a_1 a_2} Q_{a_2}^j \quad k = 0, 1, 2 \\
& [2, 0, \dots, 0] \\
\text{Odd adjoint mesons :} \quad & s^{2k+1} t^2 \rightarrow (A_{2k+1})^{ij} = Q_{a_1}^i (\phi^{2k+1})_{a_1 a_2} Q_{a_2}^j \quad k = 0, 1, 2 \\
& [0, 1, 0, \dots, 0] \\
\text{3-baryons :} \quad & t^3 \rightarrow B^{ijk} = f^{abc} Q_a^i Q_b^j Q_c^k \\
& [0, 0, 1, 0, \dots, 0] \\
\text{Adjoint 3-baryons :} \quad & st^3 \rightarrow \mathcal{B}_{0,0,1}^{i_1 i_2 j_1} = f^{a_1 a_2 b_1} Q_{a_1}^{i_1} Q_{a_2}^{i_2} (P_1)_{b_1}^{j_1} \\
& [1, 1, 0, \dots, 0] \\
& s^2 t^3 \rightarrow \mathcal{B}_{0,0,2}^{i_1 i_2 j_1} = f^{a_1 a_2 b_1} Q_{a_1}^{i_1} Q_{a_2}^{i_2} (P_2)_{b_1}^{j_1}, \quad \mathcal{B}_{0,1,1}^{i_1 j_1 j_2} = f^{a_1 b_1 b_2} Q_{a_1}^{i_1} (P_1)_{b_1}^{j_1} (P_1)_{b_2}^{j_2}, \\
& \mathcal{B}_{1,1,1}^{i_1 i_2 i_3} = \epsilon^{a_1 \dots a_7} Q_{a_1}^{i_1} Q_{a_2}^{i_2} Q_{a_3}^{i_3} \phi_{a_4 a_5} \phi_{a_6 a_7} \\
& [1, 1, 0, \dots, 0] + [0, 0, 1, 0, \dots, 0] \\
& s^3 t^3 \rightarrow \mathcal{B}_{0,0,3}^{i_1 i_2 j_1} = f^{a_1 a_2 b_1} Q_{a_1}^{i_1} Q_{a_2}^{i_2} (P_3)_{b_1}^{j_1}, \quad \mathcal{B}_{0,1,2}^{i_1 j_1 k_1} = f^{a_1 a_2 a_3} Q_{a_1}^{i_1} (P_1)_{b_1}^{j_1} (P_2)_{c_1}^{k_1}, \\
& \mathcal{B}_{1,1,1}^{i_1 i_2 i_3} = f^{a_1 a_2 a_3} (P_1)_{a_1}^{i_1} (P_1)_{a_2}^{i_2} (P_1)_{a_3}^{i_3} \\
& [3, 0, \dots, 0] + [0, 0, 1, 0, \dots, 0] \\
& s^4 t^3 \rightarrow \mathcal{B}_{0,0,4} = f Q Q P_4, \quad \mathcal{B}_{0,1,3} = f Q P_1 P_3, \quad \mathcal{B}_{0,2,2} = f Q P_2 P_2, \\
& \mathcal{B}_{1,1,2} = f P_1 P_1 P_2 \\
& [1, 1, 0, \dots, 0] \\
& s^5 t^3 \rightarrow \mathcal{B}_{0,0,5} = f Q Q P_5, \quad \mathcal{B}_{0,1,4} = f Q P_1 P_4, \quad \mathcal{B}_{0,2,3} = f Q P_2 P_3, \\
& \mathcal{B}_{1,2,2} = f P_1 P_2 P_2, \quad \mathcal{B}_{1,1,3} = f P_1 P_1 P_3 \\
& [1, 1, 0, \dots, 0] \\
& s^6 t^3 \rightarrow \mathcal{B}_{0,0,6} = f Q Q P_6, \quad \mathcal{B}_{0,1,5} = f Q P_1 P_5, \quad \mathcal{B}_{0,2,4} = f Q P_2 P_4, \\
& \mathcal{B}_{0,3,3} = f Q P_3 P_3, \quad \mathcal{B}_{1,2,3} = f P_1 P_2 P_3, \quad \mathcal{B}_{2,2,2} = f P_2 P_2 P_2, \\
& \mathcal{B}_{1,1,4} = f P_1 P_1 P_4 \\
& [3, 0, \dots, 0] \\
\text{4-baryons :} \quad & t^4 \rightarrow b^{i_1 \dots i_4} = \epsilon^{a_1 \dots a_7} Q_{a_1}^{i_1} \dots Q_{a_4}^{i_4} f_{a_5 a_6 a_7} \\
& [0, 0, 0, 1, 0, \dots, 0] \\
\text{Adjoint 4-baryons :} \quad & st^4 \rightarrow \mathfrak{b}_{0,0,0,1}^{i_1 \dots i_4} = \epsilon^{a_1 \dots a_7} Q_{a_1}^{i_1} Q_{a_2}^{i_2} Q_{a_3}^{i_3} (P_1)_{a_4}^{i_4} f_{a_5 a_6 a_7} \\
& [1, 0, 1, 0, \dots, 0] \\
& s^2 t^4 \rightarrow \mathfrak{b}_{0,0,0,2}^{i_1 \dots i_4} = \epsilon^{a_1 \dots a_7} Q_{a_1}^{i_1} Q_{a_2}^{i_2} Q_{a_3}^{i_3} (P_2)_{a_4}^{i_4} f_{a_5 a_6 a_7}, \\
& \mathfrak{b}_{0,0,1,1}^{i_1 \dots i_4} = \epsilon^{a_1 \dots a_7} Q_{a_1}^{i_1} Q_{a_2}^{i_2} (P_1)_{a_3}^{i_3} (P_1)_{a_4}^{i_4} f_{a_5 a_6 a_7} \\
& [1, 0, 1, 0, \dots, 0] + [0, 2, 0, \dots, 0] \\
& s^3 t^4 \rightarrow \mathfrak{b}_{0,0,1,2}^{i_1 \dots i_4} = \epsilon^{a_1 \dots a_7} Q_{a_1}^{i_1} Q_{a_2}^{i_2} (P_1)_{a_3}^{i_3} (P_2)_{a_4}^{i_4} f_{a_5 a_6 a_7}, \\
& \mathfrak{b}_{0,1,1,1}^{i_1 \dots i_4} = \epsilon^{a_1 \dots a_7} Q_{a_1}^{i_1} (P_1)_{a_2}^{i_2} (P_1)_{a_3}^{i_3} (P_1)_{a_4}^{i_4} f_{a_5 a_6 a_7} \\
& [2, 1, 0, \dots, 0]
\end{aligned}$$

Note that the Casimir invariant $u_4 \equiv \text{Tr}(\phi^4)$ can be obtained from u_2 as follows:

$$u_4 = \frac{1}{4}u_2^2 . \quad (7.5)$$

Therefore, we do not include u_4 in the above list.

As for the case of the $SU(N_c)$ gauge groups, we emphasise that the representations written above are *not* the ones in which the generators transform; however, they are the ones in which the relations have already been taken into account.

Total number of generators. Using the trick mentioned in Section 4.1.2, we find that the total number of generators is $\frac{1}{12} \left(24 - 524N_f + 957N_f^2 - 430N_f^3 + 69N_f^4 \right)$.

8. A Geometric Aperçu

In the preceding sections, we computed Hilbert series and their plethystic logarithms which count generators and relations in the chiral rings of adjoint SQCD. In the following, we shall extract a number of useful geometrical properties of moduli spaces from Hilbert series.

8.1 Palindromic Numerator

We have observed in many case studies before that the numerator of the Hilbert series for adjoint SQCD is palindromic, *i.e.* it can be written in the form of a degree k_1, k_2 polynomial in s, t :

$$P_{k_1, k_2}(s, t) = \sum_{m=0}^{k_1} \sum_{n=0}^{k_2} a_{m, n} s^m t^n , \quad (8.1)$$

with symmetric coefficients $a_{(k_1-m), (k_2-n)} = a_{m, n}$. We establish the following theorem:

Theorem 8.1. *Let $P(s, t)$ be a numerator of the Hilbert series $g(s, t)$ such that $P(1, 1) \neq 0$. Then, $P(s, t)$ is palindromic.*

We shall use the following lemma to prove the above theorem.

Lemma 8.2. *Let d be the dimension of the moduli space and let d_G be the dimension of the gauge group G . Then, the Hilbert series obey:*

$$g^{(N_f, G)}(1/s, 1/t) = (-1)^d s^{d_G} t^d g^{(N_f, G)}(s, t) \quad (8.2)$$

Proof. For definiteness, we shall prove only the case of $G = SU(N_c)$, *i.e.*,

$$g^{(N_f, SU(N_c))}(1/s, 1/t) = (-1)^d s^{N_c^2-1} t^d g^{(N_f, SU(N_c))}(s, t) , \quad (8.3)$$

where $d = 2N_f N_c$. The others can be proven in a similar fashion.

We write down the Molien–Weyl formula as

$$\begin{aligned}
g^{(N_f, SU(N_c))}(s, t) &= \int_{SU(N_c)} d\mu_{SU(N_c)} \text{PE} [N_f ([1, 0, \dots, 0] + [0, \dots, 0, 1]) t + [1, 0, \dots, 0, 1] s] \\
&= \int_{SU(N_c)} \frac{d\mu_{SU(N_c)}}{\prod_w (1 - t \prod_{l=1}^{N_c-1} z_l^{w_l})^{N_f} (1 - t \prod_{l=1}^{N_c-1} z_l^{-w_l})^{N_f}} \times \frac{1}{(1-s)^{N_c-1}} \times \\
&\quad \frac{1}{\prod_{\alpha^+} \left(1 - s \prod_{l=1}^{N_c-1} z_l^{\alpha_l^+} \right) \left(1 - s \prod_{l=1}^{N_c-1} z_l^{-\alpha_l^+} \right)}, \tag{8.4}
\end{aligned}$$

where w denotes the collection of the weights of the fundamental representation, α^+ denotes the collection of the positive roots, and the subscript l denotes the l -th component of the weight or root.

Let us turn to $g^{(N_f, SU(N_c))}(1/s, 1/t)$. Under the transformations $s \rightarrow 1/s$ and $t \rightarrow 1/t$, the integrand in (8.4) changes to (up to a minus sign)

$$\frac{t^{2N_f N_c}}{\prod_w (1 - t \prod_{l=1}^{N_c-1} z_l^{w_l})^{N_f} (1 - t \prod_{l=1}^{N_c-1} z_l^{-w_l})^{N_f}} \cdot \frac{s^{N_c^2-1}}{(1-s)^{N_c-1} \prod_{\alpha^+} \left(1 - s \prod_{l=1}^{N_c-1} z_l^{\alpha_l^+} \right) \left(1 - s \prod_{l=1}^{N_c-1} z_l^{-\alpha_l^+} \right)}.$$

Now let us obtain the correct overall sign. An easy way to do so is to consider the expansion of $g^{(N_f, SU(N_c))}(s, t)$ as a Laurent series around $s, t = 1$:

$$g^{(N_f, N_c)}(s, t) = \sum_{m_1=-p}^{\infty} \sum_{m_2=-q}^{\infty} c_{m_1, m_2} (s-1)^{m_1} (t-1)^{m_2} \sim \frac{c_{-p, -q}}{(s-1)^p (t-1)^q}, \tag{8.5}$$

(as $s, t \rightarrow 1$) where $p+q=d$. (Recall that d is the dimension of the moduli space, which is equal to the order of the pole at $t=1$ of the unrefined Hilbert series $g^{(N_f, N_c)}(t, t)$.) Therefore, we see that as $s, t \rightarrow 1$, the signs of $g^{(N_f, SU(N_c))}(1/s, 1/t)$ and $g^{(N_f, SU(N_c))}(s, t)$ differ by $(-1)^d$. Combining this with (8.5), we prove the first formula in (8.3). \square

We are now ready for our claim.

Proof of Theorem 8.1. We note that the denominator of the Hilbert series $g(s, t)$ is in the form $\prod_k (1 - s^{a_k})^{b_k} \prod_l (1 - t^{c_l})^{d_l}$, where a_k, b_k, c_l, d_l are non-negative integers. Observe that upon the transformation $s \rightarrow 1/s$ and $t \rightarrow 1/t$, the denominator picks up the sign $(-1)^{\sum_k b_k + \sum_l d_l}$. Now if the numerator $P(t)$ does not vanish at $s = t = 1$, then $\sum_k b_k + \sum_l d_l$ is exactly the order of the pole at $t = 1$ of the unrefined Hilbert series $g(t, t)$, which is equal to the dimension d of the moduli space. Since $P(s, t) = g(s, t) \prod_k (1 - s^{a_k})^{b_k} \prod_l (1 - t^{c_l})^{d_l}$, it follows from (8.2) that $P(s, t)$ is palindromic. \square

8.2 The Adjoint SQCD vacuum Is Calabi-Yau

Similar situations were encountered in [1, 2, 16]. Due to a well-known theorem in commutative

algebra called the Hochster–Roberts theorem^{14,15} [30], our coordinate rings of the moduli space are Cohen–Macaulay. Therefore, as an immediate consequence of Theorem 8.1 and the Stanley theorem¹⁶ [31], the chiral rings are also algebraically Gorenstein. Since affine Gorenstein varieties means Calabi–Yau, we reach an important conclusion:

Theorem 8.3. *The moduli space of the adjoint SQCD is an affine Calabi–Yau cone over a weighted projective variety.*

8.3 The Adjoint SQCD Moduli Space Is Irreducible

In this subsection, we demonstrate that the classical moduli space of adjoint SQCD is irreducible. Similar situations were encountered in [1, 2].

As in [1], the moduli space (in the absense of a superpotential) of adjoint SQCD can be described by a symplectic quotient:

$$\mathbb{C}^n // G = \mathbb{C}^n / G^c, \quad (8.6)$$

where G^c denotes the complexification of gauge group and n is the total number of the chiral superfields (both transforming in fundamental and adjoint representations):

$$n = \begin{cases} 2N_c N_f + (N_c^2 - 1) & \text{for } G = SU(N_c) \\ 4N_c N_f + \frac{1}{2}N_c(N_c + 1) & \text{for } G = Sp(N_c) \\ N_c N_f + \frac{1}{2}N_c(N_c - 1) & \text{for } G = SO(N_c) \\ 7N_f + 14 & \text{for } G = G_2. \end{cases}$$

Since \mathbb{C}^n is irreducible and G^c is a continuous group, we expect the resulting quotient to be also irreducible¹⁷.

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¹⁴This theorem states that the invariant ring of a linearly reductive group acting on a regular ring is Cohen–Macaulay.

¹⁵We are grateful to Richard Thomas for drawing our attention to this important theorem.

¹⁶This theorem states that the numerator to the Hilbert series of a graded Cohen–Macaulay domain R is palindromic if and only if R is Gorenstein.

¹⁷We are grateful to Alberto Zaffaroni for this point.

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